Two-Dimensional Two-Component Plasma with Adsorbing Impurities

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Abstract We study the behavior of the two-dimensional two-component plasma in the presence of some adsorbing impurities. Using a solvable model, we find analytic expressions for the thermodynamic properties of the plasma such as the n-body densities, the grand potential, and the pressure. We specialize in the case where there are one or two adsorbing point impurities in the plasma, and in the case where there are one or two parallel adsorbing lines. In the former case we study the effective interaction between the impurities, due to the charge redistribution around them. The latter case is a model for electrodes with adsorbing sticky sites on their surface.

Keywords Coulomb gas · Adsorption sites

1 Introduction

A two-component plasma (TCP) is a system composed of two different types of particles with charges $\pm e$ interacting through a Coulomb potential. In this work, we study a two-dimensional model which is exactly solvable. The solution of Poisson equation shows that in two dimensions, the Coulomb interaction between two particles, with charges q_1, q_2, at a distance r from each other, is

$$v(r) = -q_1 q_2 \ln(r/L), \tag{1.1}$$

where L is an irrelevant length scale fixing the zero of the potential.

Let us define the dimensionless coupling constant $\Gamma = \beta e^2$, where $\beta = (k_B T)^{-1}$, T the temperature and k_B the Boltzmann constant. The coupling constant Γ is the ratio between the electrostatic and thermal energy of the particles of the gas. If $\Gamma < 2$, the thermal energy

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is high enough to prevent the collapse of particles of different sign. If $\Gamma \geq 2$, the system is unstable against the collapse and the point-particles must be replaced by hard disks of diameter a. Using the analogy between the two-component plasma and the sine-Gordon quantum field theory, exact results for the thermodynamic properties of the Coulomb gas, in the bulk [2] and near a planar interface [3, 4], have been obtained when $\Gamma < 2$. For $\Gamma = 2$, the classical two-dimensional two-component plasma is equivalent to a quantum free Fermi gas [1]. This fact allows us to find analytic expressions for the thermodynamic properties and correlation functions of the plasma, for a wider variety of geometries [1, 5, 6], and even in the presence of external fields [7]. In this work, we restrict our analysis to the case $\Gamma = 2$.

Assuming $\Gamma = 2$, we find analytical solutions for the thermodynamics of the two-component plasma in the presence of some impurities. Results have been obtained for a similar problem, the one-component plasma with adsorbing impurities [8, 9]. For the two-component plasma, some similar problems to the present one, with adsorbing boundaries, were considered in [10, 11].

An impurity in our model can be understood as a particle, different from the charged particles of the gas. It interacts with the particles of the gas through a non-electrical potential $U_{\pm}(\mathbf{r})$. As usual, for inhomogeneous systems under the action of an external field derived from a potential $U_{\pm}(\mathbf{r})$, we can define a position depend fugacity due to the external potential created by the impurity $m_{\pm}(\mathbf{r}) = me^{-\beta U_{\pm}(\mathbf{r})}$, where m is the bulk fugacity. Here we are interested in adsorbing impurities, which can be modeled with a position depend fugacity $m_{\pm}(\mathbf{r}) = m[1 + \alpha_{\pm}\delta(\mathbf{r} - \mathbf{R})]$, where $\alpha_{\pm}$ is the magnitude of the interaction of the impurity with the positive and the negative particles of the plasma (it will be called adhesivity), while $\mathbf{R}$ is the position of the impurity in the plasma. In order to prevent the collapse of the opposite charged particles, we will consider that the impurity only attracts one type of particles of the system. For instance, if the impurity only attracts the positive particles of the system, $\alpha_{+} = \alpha$ and $\alpha_{-} = 0$.

This simplified model could be applied to some systems. An example is a salt solution with positive and negative ions and porous sites, which can attract some of the ions, with different sizes for each species. The size of the porous sites can be such that they are small enough in order to avoid that the larger ions get adsorbed into the sites, while the smaller ones are allowed to be adsorbed. The same salt solution may have some "dusty-points"; the contact interaction may be generated by a chemical interaction among those points and the charged particles. An example of a chemical interaction can be presented in some atoms whose valence layer is not completely filled and they can receive electrons from an atom with a small electronegativity.

The present document is organized as follows. In Sect. 2, we briefly review the general mathematical treatment which allow us to find the thermodynamic properties and correlation functions of the two-component plasma in the presence of any external potential when $\Gamma = 2$ [1]. In Sect. 3, we consider the case where there are a few point impurities in the plasma. We concentrate specifically in the case of one or two impurities. We compute the density profiles around the impurities and the two-body correlation functions using the method of [1]. Also, we extend the treatment presented in [8] for the one-component plasma, to find general expressions for the grand potential, and the one- and two-body density functions for the two-component plasma with impurities, in terms of the same quantities for the unperturbed system. In Sect. 4, we study the two-component plasma with one or two lines of adsorbing impurities. We compute the one- and two-density functions and the grand potential using the method of [1]. A summary of the main results is presented in Sect. 5.

2 Method of Solution

To solve our specific model, we use the method introduced by Cornu and Jancovici [1]. This mathematical treatment is valid for a globally neutral two-component Coulomb gas with a coupling constant $\Gamma = 2$. Since the model is a two dimensional one, it is convenient to express the position $\mathbf{r} = (x, y)$ of the particles in complex coordinates as $z = x + iy$. As explained earlier, at $\Gamma = 2$, it is mandatory to introduce a cutoff distance a between particles to avoid the collapse between particles of opposite sign. This cutoff can be understood as a small diameter of the particles. It is shown in [1] that, in the limit $a \to 0$, the grand partition function Ξ in this scheme is given by

$$\ln \Xi = \mathrm{Tr}\left(\ln\left[\left(\partial\!\!\!/ + m_+(\mathbf{r})\frac{1 + \sigma_z}{2} + m_-(\mathbf{r})\frac{1 - \sigma_z}{2}\right)\partial\!\!\!/^{-1}\right]\right), \tag{2.1}$$

where $\partial\!\!\!/ = \sigma_x \partial_x + \sigma_y \partial_y$ is the two dimensional Dirac operator. The matrices $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. The position dependent fugacity, defined as $m_\pm(\mathbf{r}) = m e^{-\beta U_\pm(\mathbf{r})}$, takes into account an external potential $U_\pm(\mathbf{r})$ which can act differently on the positive or negative particles. The constant rescaled fugacity m is related to the chemical potential μ by $m = 2\pi e^{\beta\mu} L/a^2$. It has units of inverse distance, and it is shown in [1], that $(2m)^{-1}$ is the screening length.

The calculation of the k-body densities is reduced to find the set of Green functions $G_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2)$, with $s_1, s_2 = \pm 1$, of the operator intervening in (2.1). They are the elements of a 2×2 matrix which satisfy the differential equations

$$\begin{pmatrix} m_+(\mathbf{r}_1) & 2\partial_z \\ 2\partial_{\bar{z}} & m_-(\mathbf{r}_1) \end{pmatrix} \begin{pmatrix} G_{++}(\mathbf{r}_1, \mathbf{r}_2) & G_{+-}(\mathbf{r}_1, \mathbf{r}_2) \\ G_{-+}(\mathbf{r}_1, \mathbf{r}_2) & G_{--}(\mathbf{r}_1, \mathbf{r}_2) \end{pmatrix} = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{2.2}$$

The density $n_\pm(\mathbf{r})$ and two-body truncated density correlation function $n_\pm^{(2)T}(\mathbf{r}_1, \mathbf{r}_2)$ of the system are given by [1]

$$n_s(\mathbf{r}) = m_s(\mathbf{r}) G_{ss}(\mathbf{r}, \mathbf{r}), \tag{2.3a}$$

$$n_{s_1 s_2}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) = -m_{s_1}(\mathbf{r}) m_{s_2}(\mathbf{r}_2) G_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2) G_{s_2 s_1}(\mathbf{r}_2, \mathbf{r}_1), \tag{2.3b}$$

where T means truncated. More generally, the truncated k-body densities are given by [1]

$$n_{s_1 \ldots s_k}^{(k)T}(\mathbf{r}_1, \ldots, \mathbf{r}_k) = (-1)^{k+1} m_{s_1}(\mathbf{r}_1) \ldots m_{s_k}(\mathbf{r}_k)$$

$$\times \sum_{\mathrm{cycles}\ (i_1 \ldots i_k)} G_{s_{i_1} s_{i_2}}(\mathbf{r}_{i_1}, \mathbf{r}_{i_2}) \cdots G_{s_{i_k} s_{i_1}}(\mathbf{r}_{i_k}, \mathbf{r}_{i_1}). \tag{2.3c}$$

The Green functions satisfy the useful relations

$$G_{ss}(\mathbf{r}_1, \mathbf{r}_2) = \overline{G_{ss}(\mathbf{r}_2, \mathbf{r}_1)}, \tag{2.4a}$$

$$G_{s-s}(\mathbf{r}_1, \mathbf{r}_2) = -\overline{G_{-ss}(\mathbf{r}_2, \mathbf{r}_1)}. \tag{2.4b}$$

The grand potential $\beta\Omega = -\ln \Xi$ can be written as

$$\beta\Omega = -\sum_k \ln\left(1 + \lambda_k\right), \tag{2.5}$$

where λ_k are the eigenvalues of the system

$$\left[\slashed{\partial} - \frac{1}{\lambda} \begin{pmatrix} m_+(\mathbf{r}) & 0 \\ 0 & m_-(\mathbf{r}) \end{pmatrix} \right] \psi(\mathbf{r}) = 0, \quad \text{with } \psi(\mathbf{r}) = \begin{pmatrix} g(\mathbf{r}) \\ f(\mathbf{r}) \end{pmatrix}, \tag{2.6}$$

a two-component spinor. Equation (2.6) shows the mentioned equivalence between the two component plasma at $\Gamma = 2$ and a free fermion gas. The pressure p is given as usual by

$$\beta p = \frac{\partial \ln \Xi}{\partial A} = -\frac{1}{A}\beta\Omega, \tag{2.7}$$

in the thermodynamic limit, where Ω is the grand potential. The "volume" of the gas is replaced by its area A because we are dealing with a two-dimensional model.

The above results are valid in the limit $a \to 0$. However, in this limit the partition function and other thermodynamical quantities such as the pressure and the density are divergent. We shall compute the (divergent) dominant term as a function of the cutoff a. On the other hand, the truncated correlation functions are finite in the limit $a \to 0$.

3 The Plasma with Some Point Adsorbing Impurities

In this section, we study the two-component plasma in the presence of one or two impurities. First, we find the Green functions associated to this system at $\Gamma = 2$. This allows us to find the k-body densities of the plasma. We will distinguish two kind of impurities. They will be called "positive" impurities when they attract the positive particles of the plasma, and "negative" impurities in the opposite case. This is a small abuse of language, since the impurities do not carry any electrical charge by themselves alone.

Then we shall extend the general theory presented in [8], to find general expressions for the thermodynamic properties of the system with impurities in term of those same quantities for the unperturbed system. These last results are valid for any value of the coupling constant Γ.

3.1 Density and Correlations at $\Gamma = 2$

3.1.1 A Positive Impurity

We consider a single impurity in the plasma that only adsorbs the positive charged particles. In this case the fugacity becomes $m_+(\mathbf{r}) = m[1 + \alpha_1\delta(\mathbf{r} - \mathbf{R}_1)]$ and $m_-(\mathbf{r}) = m$. The constants α_1 and $\mathbf{R}_1$ represent the adhesivity of the impurity and its position.

First we compute the Green functions that are needed to find the density and correlations. To solve the system (2.2) for the Green functions, we assume solutions of the form $G_{s_1 s_2} = G^0_{s_1 s_2} + G^1_{s_1 s_2}$, where $G^0_{s_1 s_2}$ are the bulk solutions (solutions for an unperturbed plasma with $m_\pm(\mathbf{r}) = m$) [1]

$$G^0_{\pm\pm}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi} K_0(m|\mathbf{r}_1 - \mathbf{r}_2|), \tag{3.1a}$$

$$G^0_{-+}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m}{2\pi} \frac{(x_1 - x_2) + i(y_1 - y_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} K_1(m|\mathbf{r}_1 - \mathbf{r}_2|)$$

$$= \frac{m}{2\pi} e^{i\theta_{12}} K_1(m|\mathbf{r}_1 - \mathbf{r}_2|), \tag{3.1b}$$

where θ_{12} is the polar angle of the vector $\mathbf{r}_1 - \mathbf{r}_2$ and $K_0(x)$, $K_1(x)$ are the modified Bessel functions of second kind of order 0 and 1. The bulk solutions satisfy the appropriate boundary condition at $\mathbf{r}_1 = \mathbf{r}_2$ imposed by the $\delta(\mathbf{r}_1 - \mathbf{r}_2)$ in (2.2). The functions $G^1_{s_1 s_2}$ take into account the contribution of the impurity. They satisfy the differential equations

$$mG^1_{++}(\mathbf{r}_1, \mathbf{r}_2) + 2\partial_{z_1} G^1_{-+}(\mathbf{r}_1, \mathbf{r}_2)$$
$$+ m\alpha_1 \delta(\mathbf{r}_1 - \mathbf{R}_1)\big[G^0_{++}(\mathbf{r}_1, \mathbf{r}_2) + G^1_{++}(\mathbf{r}_1, \mathbf{r}_2)\big] = 0, \tag{3.2a}$$

$$2\partial_{\bar{z}_1} G^1_{++}(\mathbf{r}_1, \mathbf{r}_2) + mG^1_{-+}(\mathbf{r}_1, \mathbf{r}_2) = 0, \tag{3.2b}$$

$$mG^1_{+-}(\mathbf{r}_1, \mathbf{r}_2) + 2\partial_{z_1} G_{--}(\mathbf{r}_1, \mathbf{r}_2)$$
$$+ m\alpha_1 \delta(\mathbf{r}_1 - \mathbf{R}_1)\big[G^0_{+-}(\mathbf{r}_1, \mathbf{r}_2) + G^1_{+-}(\mathbf{r}_1, \mathbf{r}_2)\big] = 0, \tag{3.2c}$$

$$2\partial_{\bar{z}_1} G^1_{+-}(\mathbf{r}_1, \mathbf{r}_2) + mG^1_{--}(\mathbf{r}_1, \mathbf{r}_2) = 0. \tag{3.2d}$$

We will first solve the equations for G^1_{++} and G^1_{-+}. We divide the space into two regions: $r_1 < R_1$ and $r_1 > R_1$. These regions will be denoted by the superscripts (1) and (2) respectively. The general solutions for $G^1_{\pm +}$ are

$$G^1_{++}(\mathbf{r}_1, \mathbf{r}_2) = \begin{cases} \sum_{l \in \mathbb{Z}} e^{il\theta_1} B^{(1)}_l I_l(mr_1), & r_1 < R_1, \\ \sum_{l \in \mathbb{Z}} e^{il\theta_1} A^{(2)}_l K_l(mr_1), & r_1 > R_1, \end{cases} \tag{3.3a}$$

$$G^1_{-+}(\mathbf{r}_1, \mathbf{r}_2) = \begin{cases} -\sum_{l \in \mathbb{Z}} e^{il\theta_1} B^{(1)}_{l-1} I_l(mr_1), & r_1 < R_1, \\ \sum_{l \in \mathbb{Z}} e^{il\theta_1} A^{(2)}_{l-1} K_l(mr_1), & r_1 > R_1, \end{cases} \tag{3.3b}$$

with $I_l(x)$ and $K_l(x)$ the modified Bessel functions of first and second kind of order l. From (3.2a), we notice that G^1_{++} is continuous, but that G^1_{-+} must be discontinuous at $\mathbf{r}_1 = \mathbf{R}_1$, due to the $\delta(\mathbf{r}_1 - \mathbf{R}_1)$ term. Multiplying (3.2a) by $e^{-il\theta_1}$ and integrating on $\mathbf{r}_1$ in a small annulus domain, centered at the origin and containing $\mathbf{R}_1$, gives

$$G^1_{++}(\mathbf{r}_1, \mathbf{r}_2)|_{r_1 = R_1^+} = G^1_{++}(\mathbf{r}_1, \mathbf{r}_2)|_{r_1 = R_1^-}, \tag{3.4a}$$

$$\frac{m\alpha_1}{2\pi R_1}\big[G^0_{++}(\mathbf{R}_1, \mathbf{r}_2) + G^1_{++}(\mathbf{R}_1, \mathbf{r}_2)\big]$$
$$= -\big[G^{(1)l+1}_{-+}(\mathbf{r}_1, \mathbf{r}_2)|_{r_1 = R_1^+} - G^{(1)l+1}_{-+}(\mathbf{r}_1, \mathbf{r}_2)|_{r_1 = R_1^-}\big], \tag{3.4b}$$

where $G^{(1)l+1}_{-+}$ refers to the $l + 1$ term of the sum in (3.3b). Imposing the boundary conditions (3.4), we find

$$A^{(2)}_l = B^{(1)}_l \frac{I_l(mR_1)}{K_l(mR_1)} \tag{3.5}$$

and

$$B^{(1)}_l = -\frac{m^3 \alpha_1 K_l(mR_1) e^{-il\theta_{R_1}} K_0(m|\mathbf{R}_1 - \mathbf{r}_2|)}{4\pi^2 [1 + \frac{m^2\alpha_1}{2\pi} \sum_{n \in \mathbb{Z}} K_n(mR_1) I_n(mR_1)]}, \tag{3.6}$$

where θ_{R_1} is the polar angle of the vector $\mathbf{R}_1$.

At this point, let us comment a few details about the bulk density, for an unperturbed system, which will be useful to interpret the denominator in (3.6). The bulk density is

given by $n_\pm^0(\mathbf{r}) = mG_{\pm\pm}^0(\mathbf{r}, \mathbf{r})$. Nevertheless, this expression diverges, since, for small argument, $K_0(x) \sim \ln(2/x) - C$, with $C \simeq 0.5772$ the Euler constant [1]. This divergence can be avoided by replacing the point particles by hard spheres of size a, which represents the minimum distance between two charged particles. Then, we compute the bulk density as $n_\pm^0 = mG_{\pm\pm}^0(\mathbf{r}_1, \mathbf{r}_2)$ with $|\mathbf{r}_1 - \mathbf{r}_2| = a \to 0$. Thus,

$$n_\pm^0 = n_0 = \frac{m^2}{2\pi} K_0(m|\mathbf{r}_1 - \mathbf{r}_2|) = \frac{m^2}{2\pi} K_0(ma) \simeq \frac{m^2}{2\pi} \left[\ln \frac{2}{ma} - C \right], \qquad (3.7)$$

where we used the small-argument expansion of the Bessel function K_0. Now, let us recall the expansion $K_0(m|\mathbf{r}_1 - \mathbf{r}_2|) = \sum_{n \in \mathbb{Z}} e^{in(\theta_1 - \theta_2)} I_n(mr_<) K_n(mr_>)$ with $r_< = \min(\mathbf{r}_1, \mathbf{r}_2)$ and $r_> = \max(\mathbf{r}_1, \mathbf{r}_2)$ [12]. With this expansion, we can obtain another formal (divergent) expression for the bulk density

$$n_\pm^0(\mathbf{R}_1) = n_0 = \frac{m^2}{2\pi} \sum_{n \in \mathbb{Z}} K_n(mR_1) I_n(mR_1). \qquad (3.8)$$

Since the unperturbed system is homogeneous the bulk density is constant, $n_\pm^0(\mathbf{R}_1) = n_0$ does not depend on $\mathbf{R}_1$. We notice that it is this expression that precisely appears in the denominator of (3.6). Then,

$$B_l^{(1)} = -\frac{m^3 \alpha_1 K_l(mR_1) e^{-il\theta_{R_1}} K_0(m|\mathbf{R}_1 - \mathbf{r}_2|)}{4\pi^2(1 + \alpha_1 n_0)}. \qquad (3.9)$$

Replacing in (3.3a) and (3.3b),

$$G_{++}^1(\mathbf{r}_1, \mathbf{r}_2) = -\frac{m^3 \alpha_1}{4\pi^2[1 + \alpha_1 n_0]} K_0(m|\mathbf{R}_1 - \mathbf{r}_2|) K_0(m|\mathbf{r}_1 - \mathbf{R}_1|), \qquad (3.10a)$$

$$G_{-+}^1(\mathbf{r}_1, \mathbf{r}_2) = -\frac{m^3 \alpha_1 e^{i\theta_{1R_1}}}{4\pi^2[1 + \alpha_1 n_0]} K_0(m|\mathbf{r}_2 - \mathbf{R}_1|) K_1(m|\mathbf{r}_1 - \mathbf{R}_1|), \qquad (3.10b)$$

with θ_{1R_1} the polar angle of the vector $\mathbf{r}_1 - \mathbf{R}_1$.

In a similar way, the other two Green functions are

$$G_{--}^1(\mathbf{r}_1, \mathbf{r}_2) = \frac{m^3 \alpha_1 e^{i(\theta_{1R_1} - \theta_{2R_1})}}{4\pi^2[1 + \alpha_1 n_0]} K_1(m|\mathbf{r}_1 - \mathbf{R}_1|) K_1(m|\mathbf{r}_2 - \mathbf{R}_1|), \qquad (3.11a)$$

$$G_{+-}^1(\mathbf{r}_1, \mathbf{r}_2) = \frac{m^3 \alpha_1 e^{-i\theta_{2R_1}}}{4\pi^2[1 + \alpha_1 n_0]} K_0(m|\mathbf{r}_1 - \mathbf{R}_1|) K_1(m|\mathbf{R}_1 - \mathbf{r}_2|), \qquad (3.11b)$$

with θ_{2R_1} the polar angle of the vector $\mathbf{r}_2 - \mathbf{R}_1$. The one-body densities can be calculated by using (2.3a), the result is

$$n_+(\mathbf{r}) = \left[1 + \alpha_1 \delta(\mathbf{r} - \mathbf{R}_1)\right] \left[n_0 - \left(\frac{m^2}{2\pi}\right)^2 \frac{\alpha_1 [K_0(m|\mathbf{R}_1 - \mathbf{r}|)]^2}{[1 + \alpha_1 n_0]}\right], \qquad (3.12a)$$

$$n_-(\mathbf{r}) = n_0 + \left(\frac{m^2}{2\pi}\right)^2 \frac{\alpha_1 [K_1(m|\mathbf{R}_1 - \mathbf{r}|)]^2}{[1 + \alpha_1 n_0]}. \qquad (3.12b)$$

The number of positive and negative particles of the plasma, $N_\pm = \int n_\pm(\mathbf{r}) \, d^2\mathbf{r}$, is

$$N_+ = N_- = n_0 \pi R^2 - \frac{m^2 \alpha_1}{4\pi [1 + \alpha_1 n_0]} + \frac{n_0 \alpha_1}{1 + \alpha_1 n_0}, \tag{3.13}$$

where we supposed that the plasma is confined in a large disk of radius R.

The difference $N_+ - N_- = 0$. This shows that the global neutrality of the system is not changed, only the charge distribution. The amount of positive charge bound to the impurity, $N_+^{\alpha_1}$, is given by the term which multiply the delta distribution in (3.12a), evaluated at $\mathbf{R}_1$. The result is

$$N_+^{\alpha_1} = \frac{\alpha_1 n_0}{1 + \alpha_1 n_0}, \tag{3.14}$$

which can be interpreted as a mean occupation number or probability that the adsorbing point is occupied by a positive charge. As the adhesivity α_1 is increased, the average number of adsorbed positive particles $N_+^{\alpha_1}$ increases, as expected. In the limit $\alpha_1 \to \infty$, the maximum value of adsorbed particles is obtained $N_+^\infty = 1$. Notice that in average there cannot be more than one adsorbed particle per impurity site. Once a particle is adsorbed, any other particle of the same sign feels a strong electrostatic repulsion that prevent it from approaching the adsorption site.

This localized positive adsorbed charge is the responsible of the change in the charge distribution. Since the bound charge is positive, it repels the positive particles and attracts the negative ones. As a consequence, the density of the negative particles around the impurity increases, and the density of the positive particles around the impurity decreases. As it is expected, the larger the adhesivity α_1 is, the larger this effect is. This behavior can be seen in Figs. 1, 2 and 3 which show the density profiles of the positive and negative particles, and the charge density profile.

If the impurity attracts the negative particles, instead of the positive ones, we obtain the same results by interchanging $(+ \longleftrightarrow -)$.

As seen in Figs. 1, 2 and 3, a charge polarization cloud is formed around the impurity. It is interesting to evaluate the electric potential created by this charge distribution. To compute the electric potential, we use the Poisson equation $\nabla^2 \phi(\mathbf{r}) = -2\pi e [n_+(\mathbf{r}) - n_-(\mathbf{r})]$, where $n_\pm(\mathbf{r})$ are given in (3.12). In the thermodynamic limit the plasma is large enough and we can assume an angular symmetry around the impurity (locating it at the origin). Thus, Poisson


Fig. 1 Density profile of positive particles for $m^2 \alpha_1 = 1, 0.5$ and 0.2 from the *darkest* to the *lightest*. The *dashed line* represents the bulk density. The vertical line at $mr = 1$ represents the Dirac distribution related to the impurity position which is located at $mR_1 = 1$. The angular difference is $\theta - \theta_{R_1} = 0$. The cutoff is $ma = 0.01$
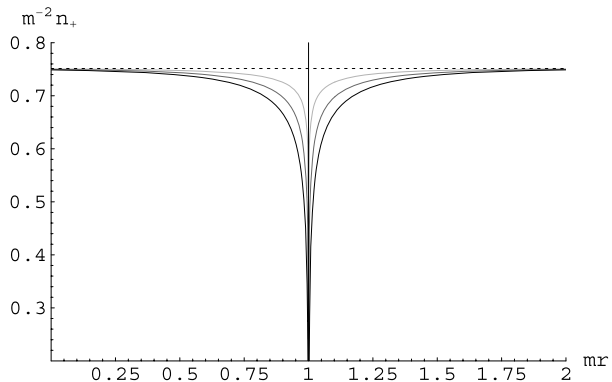
Fig. 2 Density profile of negative particles for $m^2\alpha_1 = 1, 0.5$ and 0.2 from the *darkest* to the *lightest*, with the same parameters as in Fig. 1
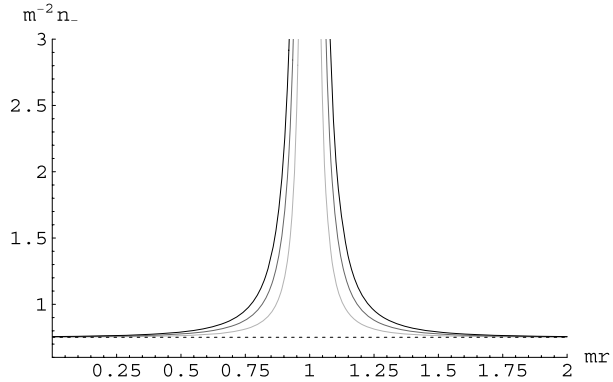
Fig. 3 Charge density profile for $m^2\alpha_1 = 1, 0.5$ and 0.2 from the *darkest* to the *lightest*, with the same parameters as in Fig. 1
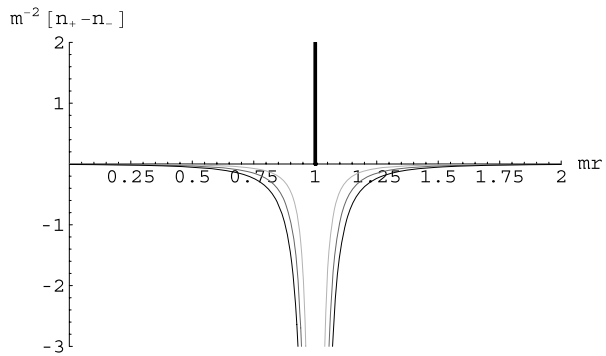
equation reduces to

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\phi(r)\right] = -2\pi e[n_+(r) - n_-(r)]. \tag{3.15}$$

Integrating, we find the electric potential

$$\phi(r) = \frac{em^2\alpha_1}{2\pi[1+\alpha_1 n_0]}[K_0(mr)]^2 = \frac{em^2 N_+^{\alpha_1}}{2n_0\pi}[K_0(mr)]^2. \tag{3.16}$$

From (3.16) we can see that close to the impurity, the potential behaves as $[\ln(mr)]^2$, which is stronger than the bare Coulomb potential $-\ln(r/L)$. This is due to the strong electrostatic coupling considered here [18], for smaller coupling, one would expect that the electric potential close to the impurity behaves as the bare Coulomb potential. Far away from it, the interaction decays exponentially due to the screening effect.

3.1.2 Two Positive Impurities

Now, we consider two impurities located at $\mathbf{R}_1$ and $\mathbf{R}_2$ with adhesivities α_1 and α_2 respectively, both attracting positive particles. The fugacities are now given by $m_+(\mathbf{r}) = m[1 + \alpha_1\delta(\mathbf{r} - \mathbf{R}_1) + \alpha_2\delta(\mathbf{r} - \mathbf{R}_2)]$ and $m_-(\mathbf{r}) = m$.

To solve the system of (2.2) for the Green functions, we assume they are of the form $G_{s_1 s_2} = G_{s_1 s_2}^0 + G_{s_1 s_2}^1 + G_{s_1 s_2}^2$ where $G_{s_1 s_2}^1$ are the solutions previously found for one impurity

located at $\mathbf{R}_1$. Without lost of generality we choose $R_2 > R_1$. The space is now divided into three regions. The first region is $r_1 < R_1$, the second region is $R_1 < r_1 < R_2$ and the third one is $R_2 < r_1$. The differential equations satisfied by $G^2_{\pm+}$ are

$$m\left[1 + \alpha_1\delta(\mathbf{r}_1 - \mathbf{R}_1) + \alpha_2\delta(\mathbf{r}_1 - \mathbf{R}_2)\right]G^2_{++}(\mathbf{r}_1, \mathbf{r}_2)$$

$$+ m\alpha_2\delta(\mathbf{r}_1 - \mathbf{R}_2)\left[G^0_{++}(\mathbf{r}_1, \mathbf{r}_2) + G^1_{++}(\mathbf{r}_1, \mathbf{r}_2)\right] + 2\partial_{z_1}G^2_{-+}(\mathbf{r}_1, \mathbf{r}_2) = 0 \quad (3.17a)$$

$$2\partial_{\bar{z}_1}G^2_{++}(\mathbf{r}_1, \mathbf{r}_2) + mG^2_{-+}(\mathbf{r}_1, \mathbf{r}_2) = 0, \tag{3.17b}$$

with general solutions of the form

$$G^2_{++}(\mathbf{r}_1, \mathbf{r}_2) = \begin{cases} \sum_{l\in\mathbb{Z}} e^{il\theta_1} B^{(1)}_l I_l(mr_1), & r_1 < R_1, \\ \sum_{l\in\mathbb{Z}} e^{il\theta_1} [B^{(2)}_l I_l(mr_1) + A^{(2)}_l K_l(mr_1)], & R_1 < r_1 < R_2, \\ \sum_{l\in\mathbb{Z}} e^{il\theta_1} A^{(3)}_l K_l(mr_1), & R_2 < r_1, \end{cases}$$

$$\tag{3.18}$$

$$G^2_{-+}(\mathbf{r}_1, \mathbf{r}_2) = \begin{cases} -\sum_{l\in\mathbb{Z}} e^{il\theta_1} B^{(1)}_{l-1} I_l(mr_1), & r_1 < R_2, \\ \sum_{l\in\mathbb{Z}} e^{il\theta_1} [-B^{(2)}_{l-1} I_l(mr_1) + A^{(2)}_{l-1} K_l(mr_1)], & R_1 < r_1 < R_2, \\ \sum_{l\in\mathbb{Z}} e^{il\theta_1} A^{(3)}_{l-1} K_l(mr_1), & R_2 < r_1. \end{cases}$$

The delta functions in (3.17a) impose that G^2_{++} is continuous at $\mathbf{r}_1 = \mathbf{R}_1$ and $\mathbf{r}_1 = \mathbf{R}_2$ but G^2_{-+} is discontinuous. Imposing these four boundary conditions, we find after some algebra

$$A^{(2)}_l = -\frac{m^4\alpha_1\alpha_2 K_0(m|\mathbf{R}_1 - \mathbf{R}_2|)}{4\pi^2\eta} I_l(mR_1)e^{-il\theta_{R_1}} D_1,$$

$$A^{(3)}_l = \frac{m^2\alpha_2}{2\pi\eta} I_l(mR_2)e^{-il\theta_{R_2}}[1 + \alpha_1 n_0]D_1 + A^{(2)}_l,$$

$$\tag{3.19}$$

$$B^{(2)}_l = \frac{m^2\alpha_2}{2\pi\eta} K_l(mR_2)e^{-il\theta_{R_2}}[1 + \alpha_1 n_0]D_1,$$

$$B^{(1)}_l = B^{(2)}_l - \frac{m^4\alpha_1\alpha_2 K_0(m|\mathbf{R}_1 - \mathbf{R}_2|)}{4\pi^2\eta} K_l(mR_1)e^{-il\theta_{R_1}} D_1,$$

with

$$D_1 = -\frac{m}{2\pi} K_0(m|\mathbf{r}_2 - \mathbf{R}_2|) + \frac{m^3\alpha_1 K_0(m|\mathbf{R}_1 - \mathbf{r}_2|)K_0(m|\mathbf{R}_1 - \mathbf{R}_2|)}{4\pi^2[1 + \alpha_1 n_0]}, \tag{3.20}$$

$$\eta = 1 + \alpha_1 n_0 + \alpha_2 n_0 + \alpha_1\alpha_2 n_0^2 - \frac{m^4\alpha_1\alpha_2}{4\pi^2}\left[K_0(m|\mathbf{R}_1 - \mathbf{R}_2|)\right]^2. \tag{3.21}$$

The functions $G^2_{\pm+}$ are therefore

$$G^2_{++}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m^2\alpha_2}{2\pi\eta} D_1\left[[1 + \alpha_1 n_0]K_0(m|\mathbf{r}_1 - \mathbf{R}_2|)\right.$$

$$\left. - \frac{m^2\alpha_1}{2\pi} K_0(m|\mathbf{r}_1 - \mathbf{R}_1|)K_0(m|\mathbf{r}_1 - \mathbf{R}_2|)\right], \tag{3.22a}$$

Fig. 4 Charge density profile for two "positive" impurities. The adhesivities are $m^2\alpha_1 = 0.2$ and $m^2\alpha_2 = 0.5$. The positions of the impurities are on the x axis at $mR_1 = 1$, $mR_1 = 1.4$ and $mR_2 = 2$, $mR_2 = 1.6$, respectively from the darkest to the lightest. All the angular differences are zero. The cutoff is $ma = 0.01$
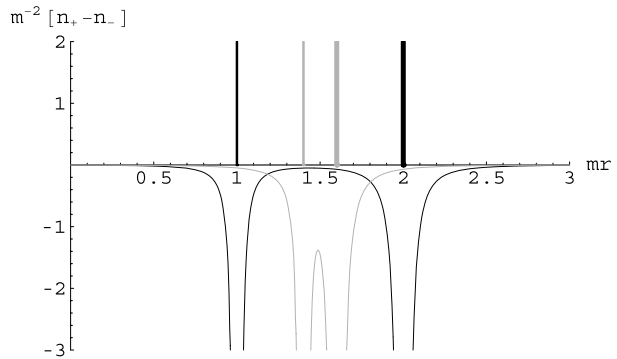
$$G^2_{-+}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m^2\alpha_2}{2\pi\eta} D_1\left[[1 + \alpha_1 n_0]K_1(m|\mathbf{r}_1 - \mathbf{R}_2|)e^{i\theta_{1R_2}}\right.$$

$$\left. - \frac{m^2\alpha_1}{2\pi} K_1(m|\mathbf{r}_1 - \mathbf{R}_1|)e^{i\theta_{1R_1}} K_0(m|\mathbf{R}_1 - \mathbf{R}_2|)\right]. \qquad (3.22b)$$

Following some similar steps, we find the other two Green functions

$$G^2_{--}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{m^2\alpha_2}{2\pi\eta} D_2\left[[1 + \alpha_1 n_0]K_1(m|\mathbf{r}_1 - \mathbf{R}_2|)e^{i\theta_{1R_2}}\right.$$

$$\left. - \frac{m^2\alpha_1}{2\pi} K_1(m|\mathbf{r}_1 - \mathbf{R}_1|)e^{i\theta_{1R_1}} K_0(m|\mathbf{R}_1 - \mathbf{R}_2|)\right], \qquad (3.23a)$$

$$G^2_{+-}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{m^2\alpha_2}{2\pi\eta} D_2\left[[1 + \alpha_1 n_0]K_0(m|\mathbf{r}_1 - \mathbf{R}_2|)\right.$$

$$\left. - \frac{m^2\alpha_1}{2\pi} K_0(m|\mathbf{r}_1 - \mathbf{R}_1|) K_0(m|\mathbf{R}_1 - \mathbf{R}_2|)\right], \qquad (3.23b)$$

with

$$D_2 = -\frac{m}{2\pi} K_1(m|\mathbf{R}_2 - \mathbf{r}_2|)e^{-i\theta_{2R_2}}$$

$$+ \frac{m^3\alpha_1 K_1(m|\mathbf{R}_1 - \mathbf{r}_2|) K_0(m|\mathbf{R}_2 - \mathbf{R}_1|)e^{-i\theta_{2R_1}}}{4\pi^2[1 + \alpha_1 n_0]}. \qquad (3.24)$$

Replacing these Green functions in (2.3a) gives the density profiles. Figure 4 shows the charge density profile. It can be seen that the charge density decreases around the adsorbing points like it was found in the problem with one impurity. The density depends on the adhesivity as well as of the distance between the impurities. An important effect arises when the adsorbing points are close enough which indicates the existence of an indirect interaction between the impurities (see Sect. 3.2.3).

3.1.3 One Positive and One Negative Impurity

Now we consider the case when the second impurity only adsorbs the negative particles. The fugacities for this configuration are $m_+(\mathbf{r}) = m[1 + \alpha_1\delta(\mathbf{r} - \mathbf{R}_1)]$ and $m_-(\mathbf{r}) = $

$m[1 + \alpha_2 \delta(\mathbf{r} - \mathbf{R}_2)]$. Assuming a similar solution as in (3.18), and after establishing the appropriate boundary conditions for this case, we solve the linear system (2.2), to find

$$
\begin{aligned}
G_{++}(\mathbf{r}_1, \mathbf{r}_2) &= G^0_{++}(\mathbf{r}_1, \mathbf{r}_2) + G^1_{++}(\mathbf{r}_1, \mathbf{r}_2) \\
&+ \frac{m^2 \alpha_2}{2\pi \bar{\eta}} D_3 \Big[[1 + \alpha_1 n_0] K_1(m|\mathbf{r}_1 - \mathbf{R}_2|) e^{-i\theta_{1 R_2}} \\
&- \frac{m^2 \alpha_1}{2\pi} K_0(m|\mathbf{r}_1 - \mathbf{R}_1|) e^{-i\theta_{R_1 R_2}} K_1(m|\mathbf{R}_1 - \mathbf{R}_2|) \Big],
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
G_{-+}(\mathbf{r}_1, \mathbf{r}_2) &= G^0_{-+}(\mathbf{r}_1, \mathbf{r}_2) + G^1_{-+}(\mathbf{r}_1, \mathbf{r}_2) \\
&- \frac{m^2 \alpha_2}{2\pi \bar{\eta}} D_3 \Big[[1 + \alpha_1 n_0] e^{i\theta_{R_2}} K_0(m|\mathbf{r}_1 - \mathbf{R}_2|) e^{-i\theta_{1 R_2}} \\
&+ \frac{m^2 \alpha_1}{2\pi} K_1(m|\mathbf{r}_1 - \mathbf{R}_1|) e^{i(\theta_{1 R_1} - \theta_{R_1 R_2})} K_1(m|\mathbf{R}_1 - \mathbf{R}_2|) \Big],
\end{aligned}
\tag{3.26}
$$

with

$$
\begin{aligned}
D_3 &= \frac{m^2}{2\pi} e^{i\theta_{2 R_2}} K_1(m|\mathbf{R}_2 - \mathbf{r}_2|) \\
&- \frac{m^3 \alpha_1 K_0(m|\mathbf{R}_1 - \mathbf{r}_2|) e^{i\theta_{R_1 R_2}} K_1(m|\mathbf{R}_2 - \mathbf{R}_1|)}{4\pi^2 [1 + \alpha_1 n_0]},
\end{aligned}
\tag{3.27}
$$

$$
\bar{\eta} = 1 + \alpha_1 n_0 + \alpha_2 n_0 + \alpha_1 \alpha_2 n_0^2 + \frac{m^4 \alpha_1 \alpha_2}{4\pi^2} \big[K_1(m|\mathbf{R}_1 - \mathbf{R}_2|) \big]^2
\tag{3.28}
$$

and $\theta_{R_1 R_2}$ the polar angle of the vector $\mathbf{R}_1 - \mathbf{R}_2$.

In order to find G_{--}, we note that if we interchange $\mathbf{R}_1$ and $\mathbf{R}_2$ and their adhesivities in the problem for G_{++}, we would have a negative impurity at $\mathbf{r}_1 = \mathbf{R}_1$, and a positive one at $\mathbf{r}_1 = \mathbf{R}_2$. Due to this symmetry argument, the function G_{--} for one positive and one negative impurity at $\mathbf{r}_1 = \mathbf{R}_1$ and $\mathbf{r}_1 = \mathbf{R}_2$, is the function G_{++} with $\mathbf{R}_1$ and $\mathbf{R}_2$ and their adhesivities interchanged. Then,

$$
G_{--}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{R}_1, \alpha_1, \mathbf{R}_2, \alpha_2) = G_{++}(\mathbf{r}_1, \mathbf{r}_2; \mathbf{R}_2, \alpha_2, \mathbf{R}_1, \alpha_1),
\tag{3.29}
$$

with G_{++} given in (3.25).

Figure 5 shows the charge density profile for this situation.

3.1.4 General Structure of the Green Functions for Two Impurities

The results of the two previous sections can be put in more general and compact form. Let $S_1 = \pm 1$ be the sign of the particles the impurity at $\mathbf{R}_1$ attracts and $S_2 = \pm 1$ the sign of the particles the second impurity at $\mathbf{R}_2$ attracts. Let us denote by $\hat{G}_{s_1 s_2} = G^0_{s_1 s_2} + G^1_{s_1 s_2}$, the Green functions for the case of only one impurity located at $\mathbf{R}_1$ ((3.10) and (3.11)). Notice that in general (3.10) and (3.11) can be written as

$$
\hat{G}_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2) = G^0_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2) - \frac{m \alpha_1}{1 + \alpha_1 n^0_{S_1}(\mathbf{R}_1)} G^0_{s_1 s_1}(\mathbf{r}_1, \mathbf{R}_1) G^0_{S_1 s_2}(\mathbf{R}_1, \mathbf{r}_2)
\tag{3.30}
$$

in terms of the Green functions $G^0_{s_1 s_2}$ and the density n^0_s of the unperturbed system.

Fig. 5 Charge density profile for one "positive" impurity and one "negative" impurity. The adhesivities are $m^2\alpha_1 = 0.2$ (*positive impurity*) and $m^2\alpha_2 = 0.5$ (*negative impurity*). The positions of the impurities are on the x axis at $mR_1 = 1$, $mR_1 = 1.4$ (*positive impurity*) and $mR_2 = 2$, $mR_2 = 1.6$ (*negative impurity*), respectively from the *darkest* to the *lightest*. All the angular differences are zero. The cutoff is $ma = 0.01$
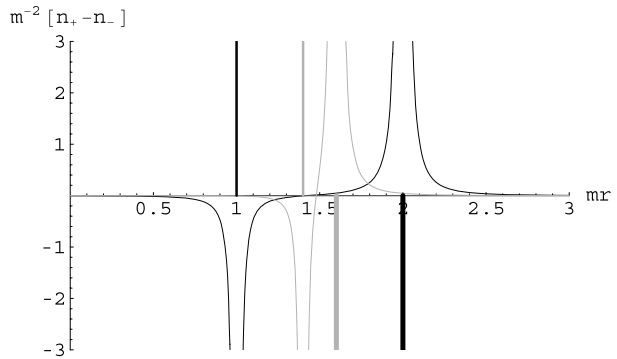
The solutions for the Green functions found in the two previous sections for the case of two impurities can be written, in term of the Green function for one impurity, as

$$G_{s_1s_2}(\mathbf{r}_1, \mathbf{r}_2) = \hat{G}_{s_1s_2}(\mathbf{r}_1, \mathbf{r}_2) - \frac{m\alpha_2}{\eta}\hat{G}_{s_1s_2}(\mathbf{r}_1, \mathbf{R}_2)\hat{G}_{S_2s_2}(\mathbf{R}_2, \mathbf{r}_2), \tag{3.31}$$

where η was defined in (3.21) and (3.28) (denoted $\bar{\eta}$ in this last equation). In general

$$\eta = 1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1) + \alpha_2 n_{S_2}^0(\mathbf{R}_2) + \alpha_1\alpha_2 n_{S_1S_2}^{0,(2)}(\mathbf{R}_1, \mathbf{R}_2), \tag{3.32}$$

where $n_{S_1S_2}^{0,(2)}(\mathbf{R}_1, \mathbf{R}_2)$ is the (non-truncated) pair correlation function for the unperturbed system. Notice the similarity in the structure of the Green functions in (3.30) and (3.31).

Replacing (3.30) into (3.31) gives the expression

$$G_{s_1s_2}(\mathbf{r}_1, \mathbf{r}_2) = G_{s_1s_2}^0(\mathbf{r}_1, \mathbf{r}_2)$$

$$- \frac{1}{\eta}\bigg[m\alpha_1\big[1 + \alpha_2 n_{S_2}^0(\mathbf{R}_2)\big]G_{s_1S_1}^0(\mathbf{r}_1, \mathbf{R}_1)G_{S_1s_2}^0(\mathbf{R}_1, \mathbf{r}_2)$$

$$+ m\alpha_2\big[1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1)\big]G_{s_1S_2}^0(\mathbf{r}_1, \mathbf{R}_2)G_{S_2s_2}^0(\mathbf{R}_2, \mathbf{r}_2)$$

$$- m^2\alpha_1\alpha_2\big[G_{s_1S_1}^0(\mathbf{r}_1, \mathbf{R}_1)G_{S_1S_2}^0(\mathbf{R}_1, \mathbf{R}_2)G_{S_2s_2}^0(\mathbf{R}_2, \mathbf{r}_2)$$

$$+ G_{s_1S_2}^0(\mathbf{r}_1, \mathbf{R}_2)G_{S_2S_1}^0(\mathbf{R}_2, \mathbf{R}_1)G_{S_1s_2}^0(\mathbf{R}_1, \mathbf{r}_2)\big]\bigg] \tag{3.33}$$

in which the exchange symmetry between $(\mathbf{R}_1, S_1)$ and $(\mathbf{R}_2, S_2)$ is manifested.

3.2 General Results

The grand potential of the system with impurities can be found by solving the eigenvalue problem (2.6) for each particular situation. We will proceed this way to compute the grand potential for the Coulomb gas in the presence of a continuous adsorbing line in Sects. 4.2.1 and 4.2.2. Although this method can also be used to compute the grand potential with point impurities [13], for this latter case, we shall proceed differently, using the general theory developed in [8] for a one-component fluid with adsorbing sites.

In the following subsections, we obtain expressions for the grand potential and the one- and two-body densities of the plasma with impurities in terms of those same quantities for

the plasma without impurities (the unperturbed system). These results are very general: they are valid for any value of the coupling Γ, they are even valid for any other type of interaction potential between the particles of the system, not only for the Coulomb system considered here. In the present case, since at $\Gamma = 2$ the grand potential and density functions for the unperturbed system are known, one can obtain explicit expressions for the thermodynamics and correlations of the plasma with impurities.

3.2.1 Grand Partition Function with an Arbitrary Number of Impurities

Suppose there are M_+ impurities attracting positive particles, located at $\mathbf{R}_1^+, \ldots, \mathbf{R}_{M_+}^+$, with adhesivities $\alpha_1^+, \ldots, \alpha_{M_+}^+$, and M_- impurities attracting negative particles, located at $\mathbf{R}_1^-, \ldots, \mathbf{R}_{M_-}^-$, with adhesivities $\alpha_1^-, \ldots, \alpha_{M_-}^-$. The general calculations done in [8] for a one-component plasma, can directly be adapted to the present two-component case, to find that the grand partition function Ξ can be written as

$$\Xi = \Xi_0 \sum_{s_+=0}^{\infty} \sum_{s_-=0}^{\infty} \sum_{k_1^+, \ldots, k_{s_+}^+=0}^{M_+} \sum_{k_1^-, \ldots, k_{s_-}^-=0}^{M_-} \frac{\prod_{n=1}^{s_+} \alpha_{k_n^+}^+ \prod_{n=1}^{s_-} \alpha_{k_n^-}^-}{s_+! s_-!}$$
$$\times n_{\{s_+\}\{s_-\}}^{0,(s_++s_-)} (\mathbf{R}_{k_1^+}^+, \ldots, \mathbf{R}_{k_{s_+}^+}^+, \mathbf{R}_{k_1^-}^-, \ldots, \mathbf{R}_{k_{s_-}^-}^-) \tag{3.34}$$

where Ξ_0 is the grand partition function of the unperturbed system and $n_{\{s_+\}\{s_-\}}^{0,(s_++s_-)}$ the $(s_+ + s_-)$-body density for s_+ positive particles and s_- negative particles of the unperturbed system. Notice that, since the correlation function vanishes if two of its arguments are equal, the above expression is not an infinite sum, it contains at most terms involving the $(M_+ + M_-)$-body correlation function and lower degree correlations.

From expression (3.34) we can obtain the density and k-body correlation functions performing successive functional derivations of the grand partition function with respect to the fugacity.

3.2.2 One Impurity

In the case of a single impurity located at $\mathbf{R}_1$, with adhesivity α_1 and attracting particles of sign S_1, (3.34) simply reduces to

$$\Xi = \left[1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1)\right] \Xi_0. \tag{3.35}$$

Then, the grand potential $\Omega = -k_B T \ln \Xi$ can be written as $\Omega = \Omega_0 + \Omega_{\text{exc}}(\alpha_1)$ with Ω_0 the grand potential of the unperturbed system and an excess grand potential

$$\beta\Omega_{\text{exc}}(\alpha_1) = -\ln\left[1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1)\right]. \tag{3.36}$$

As explained in [8], the adhesivity α can be interpreted also as a fugacity for the adsorbed particles. Thus, one can compute the average number of adsorbed particles from the relation

$$N_{S_1}^{\alpha_1} = -\alpha_1 \frac{\partial \beta\Omega_{\text{exc}}(\alpha_1)}{\partial \alpha_1} = \frac{\alpha_1 n_{S_1}^0(\mathbf{R}_1)}{1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1)}. \tag{3.37}$$

We recover the result (3.14) obtained from a direct calculation at $\Gamma = 2$.

From (3.35) we can also rederive the results from the previous section for the density and correlation functions, by performing functional derivatives of Ξ with respect to the fugacity. Remembering that

$$m_s(\mathbf{r}) \frac{\delta \ln \Xi_0}{\delta m_s(\mathbf{r})} \bigg|_{m(\mathbf{r})=m} = n_s^0(\mathbf{r}) \tag{3.38}$$

and

$$m_{s_2}(\mathbf{r}_2) \frac{\delta n_{s_1}^0(\mathbf{r}_1)}{\delta m_{s_2}(\mathbf{r}_2)} \bigg|_{m(\mathbf{r})=m} = n_{s_1 s_2}^{0,(2)T}(\mathbf{r}_1, \mathbf{r}_2), \tag{3.39}$$

deriving (3.35) with respect to $m_s(\mathbf{r})$, we obtain

$$n_{S_1}(\mathbf{r}) = \left[1 + \alpha_1 \delta(\mathbf{r} - \mathbf{R}_1)\right] \left[n_{S_1}^0(\mathbf{r}) + \frac{\alpha_1 n_{S_1 S_1}^{0,(2)T}(\mathbf{r}, \mathbf{R}_1)}{1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1)}\right] \tag{3.40a}$$

and

$$n_{-S_1}(\mathbf{r}) = n_{-S_1}^0(\mathbf{r}) + \frac{\alpha_1 n_{-S_1 S_1}^{0,(2)T}(\mathbf{r}, \mathbf{R}_1)}{1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1)}. \tag{3.40b}$$

When $\Gamma = 2$, we recover the results (3.12) from Sect. 3.1.1. Deriving (3.40) once again with respect to the fugacity, we obtain the truncated two-body correlation functions

$$n_{S_1 S_1}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) = \left[1 + \alpha_1 \delta(\mathbf{r}_1 - \mathbf{R}_1)\right]\left[1 + \alpha_1 \delta(\mathbf{r}_2 - \mathbf{R}_1)\right] n_{S_1 S_1}^{(2)T*}(\mathbf{r}_1, \mathbf{r}_2), \tag{3.41a}$$

$$n_{-S_1 S_1}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) = \left[1 + \alpha_1 \delta(\mathbf{r}_2 - \mathbf{R}_1)\right] n_{-S_1 S_1}^{(2)T*}(\mathbf{r}_1, \mathbf{r}_2), \tag{3.41b}$$

$$n_{-S_1, -S_1}^{(2)T}(\mathbf{r}_1, \mathbf{r}_2) = n_{-S_1, -S_1}^{(2)T*}(\mathbf{r}_1, \mathbf{r}_2), \tag{3.41c}$$

with

$$n_{s_1 s_2}^{(2)T*}(\mathbf{r}_1, \mathbf{r}_2) = n_{s_1 s_2}^{0,(2)T}(\mathbf{r}_1, \mathbf{r}_2) - \frac{\alpha_1^2 n_{s_1 S_1}^{0,(2)T}(\mathbf{r}_1, \mathbf{R}_1) n_{S_1 s_2}^{0,(2)T}(\mathbf{R}_1, \mathbf{r}_2)}{\left[1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1)\right]^2}$$

$$+ \frac{\alpha_1 n_{s_1 s_2 S_1}^{0,(3)T}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{R}_1)}{1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1)}, \tag{3.42}$$

where $n_{s_1 s_2 s_3}^{0,(3)T}$ is the truncated three-body density function of the unperturbed system. From the Green functions, (3.30), obtained in the previous section and the general relations (2.3), it can be shown that the correlation functions obtained in Sect. 3.1.1 are indeed those given by (3.41) and (3.42) when $\Gamma = 2$.

3.2.3 Two Impurities

Now, let us consider the case of two impurities, located at $\mathbf{R}_1$ and $\mathbf{R}_2$ with adhesivities α_1 and α_2 and attracting particles of sign S_1 and S_2, respectively. Equation (3.34) gives

$$\Xi = \Xi_0\left[1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1) + \alpha_2 n_{S_2}^0(\mathbf{R}_2) + \alpha_1 \alpha_2 n_{S_1 S_2}^{0,(2)}(\mathbf{R}_1, \mathbf{R}_2)\right] = \Xi_0 \eta \tag{3.43}$$

with η defined in (3.32). The grand potential now reads

$$\beta\Omega = \beta\Omega_0 - \ln \eta. \tag{3.44}$$

Notice that in the problems for one and two impurities the excess grand potential is the logarithm of the denominator of the Green functions $G^1_{s_1 s_2}$ and $G^2_{s_1 s_2}$ respectively. It is convenient to express the grand potential as

$$\Omega = \Omega_0 + \Omega_{\text{exc}}(\alpha_1) + \Omega_{\text{exc}}(\alpha_2) + \Omega_{S_1 S_2}(\mathbf{R}_1, \alpha_1; \mathbf{R}_2, \alpha_2), \tag{3.45}$$

with

$$\beta \Omega_{S_1 S_2}(\mathbf{R}_1, \alpha_1; \mathbf{R}_2, \alpha_2) = -\ln\left[1 + \frac{\alpha_1 \alpha_2 n^{0,(2)T}_{S_1 S_2}(\mathbf{R}_1, \mathbf{R}_2)}{[1 + \alpha_1 n^0_{S_1}(\mathbf{R}_1)][1 + \alpha_2 n^0_{S_2}(\mathbf{R}_2)]}\right]. \tag{3.46}$$

Explicitly, for $\Gamma = 2$,

$$\beta \Omega_{\pm\pm}(\mathbf{R}_1, \alpha_1; \mathbf{R}_2, \alpha_2) = -\ln\left[1 - \frac{m^4 \alpha_1 \alpha_2 [K_0(m|\mathbf{R}_1 - \mathbf{R}_2|)]^2}{4\pi^2[1 + \alpha_1 n_0][1 + \alpha_2 n_0]}\right] \tag{3.47}$$

and

$$\beta \Omega_{\pm\mp}(\mathbf{R}_1, \alpha_1; \mathbf{R}_2, \alpha_2) = -\ln\left[1 + \frac{m^4 \alpha_1 \alpha_2 [K_1(m|\mathbf{R}_1 - \mathbf{R}_2|)]^2}{4\pi^2[1 + \alpha_1 n_0][1 + \alpha_2 n_0]}\right]. \tag{3.48}$$

A work $\Omega_{\text{exc}}(\alpha_i)$ is required to introduce a single impurity into the plasma. To introduce two impurities, we require, in addition to $\Omega_{\text{exc}}(\alpha_1) + \Omega_{\text{exc}}(\alpha_2)$, an additional work $\Omega_{S_1 S_2}(\mathbf{R}_1, \alpha_1; \mathbf{R}_2, \alpha_2)$. We can define the effective chemical potential of an impurity with adhesivity α as $\mu(\alpha) = \Omega_{\text{exc}}(\alpha)$. The term $\Omega_{S_1 S_2}(\mathbf{R}_1, \alpha_1; \mathbf{R}_2, \alpha_2)$ represents an effective interaction between the impurities. This contribution is not a direct interaction between the adsorbing particles but a consequence of the inhomogeneous charge distribution that they create.

Figures 6 and 7 show the effective interaction between two adsorbing particles. It can be seen that it is repulsive if the impurities have the same sign and attractive if their sign is opposite. As expected, the interaction is important when the impurities are close enough but is negligible for large distances. It depends on the adhesivity of the two impurities, indeed, it is larger for large adhesivities.

Fig. 6 Effective interaction between two positive impurities as a function of their distance. The adhesivities are $m^2\alpha_1 = m^2\alpha_2 = 0.2$, 1 and 2 from *darkest* to *lightest*. The cutoff is $ma = 0.01$
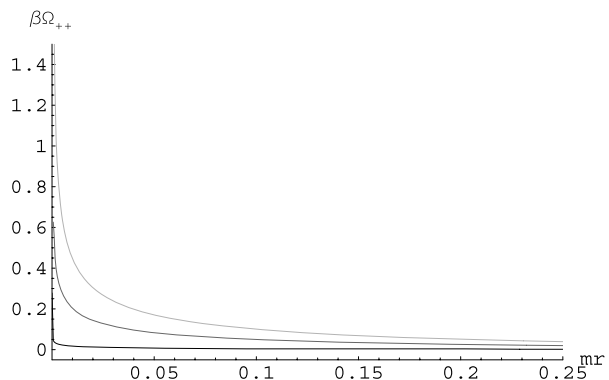
Fig. 7 Effective interaction between a positive and a negative impurity as a function of their distance. The values are the same than in Fig. 6
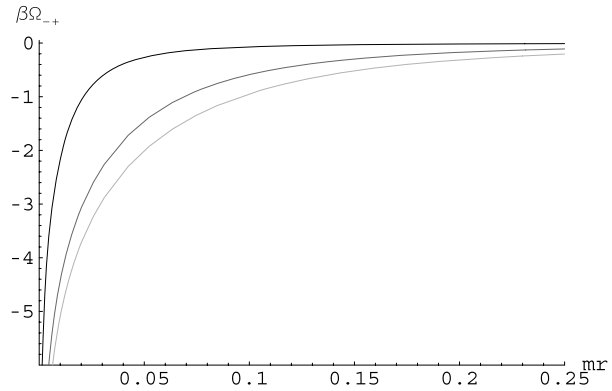
From (3.43), we can obtain the density profiles by performing a functional derivative with respect to the fugacity. We obtain

$$
n_s(\mathbf{r}) = \begin{cases} [1 + \alpha_1 \delta(\mathbf{r} - \mathbf{R}_1) + \alpha_2 \delta(\mathbf{r} - \mathbf{R}_2)] n_s^*(\mathbf{r}) & \text{if } s = S_1 = S_2, \\ [1 + \alpha_1 \delta(\mathbf{r} - \mathbf{R}_1)] n_s^*(\mathbf{r}) & \text{if } s = S_1 = -S_2, \\ [1 + \alpha_2 \delta(\mathbf{r} - \mathbf{R}_2)] n_s^*(\mathbf{r}) & \text{if } s = S_2 = -S_1, \\ n_s^*(\mathbf{r}) & \text{if } s = -S_1 = -S_2, \end{cases} \tag{3.49}
$$

with

$$
n_s^*(\mathbf{r}) = n_s^0(\mathbf{r}) + \frac{1}{\eta} \Big[\alpha_1 n_{sS_1}^{0,(2)T}(\mathbf{r}, \mathbf{R}_1) \big[1 + \alpha_2 n_{S_2}^0(\mathbf{R}_2) \big]
$$

$$
+ \alpha_2 n_{sS_2}^{0,(2)T}(\mathbf{r}, \mathbf{R}_2) \big[1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1) \big] + \alpha_1 \alpha_2 n_{sS_1S_2}^{0,(3)T}(\mathbf{r}, \mathbf{R}_1, \mathbf{R}_2) \Big]. \tag{3.50}
$$

With the Green functions (3.33) for two impurities found in Sects. 3.1.2, 3.1.3 and 3.1.4 and the general relation (2.3), it is direct to verify that the density is indeed given by (3.49) and (3.50) when $\Gamma = 2$.

We can compute the average number of adsorbed particles by each impurity $N_{S_i}^{\alpha_i}$, for $i = 1, 2$. This can be obtained either from the relation $N_{S_i}^{\alpha_i} = \alpha_i n_{S_i}^*(\mathbf{R}_i)$, or from

$$
N_{S_i}^{\alpha_i} = -\alpha_i \frac{\partial(\beta\Omega)}{\partial \alpha_i}. \tag{3.51}
$$

We obtain

$$
N_{S_i}^{\alpha_i} = \frac{\alpha_i n_{S_i}^0(\mathbf{R}_i) + \alpha_1 \alpha_2 n_{S_1 S_2}^{0,(2)}(\mathbf{R}_1, \mathbf{R}_2)}{1 + \alpha_1 n_{S_1}^0(\mathbf{R}_1) + \alpha_2 n_{S_2}^0(\mathbf{R}_2) + \alpha_1 \alpha_2 n_{S_1 S_2}^{0,(2)}(\mathbf{R}_1, \mathbf{R}_2)}. \tag{3.52}
$$

4 The Plasma with Adsorbing Lines

4.1 Density and Correlations

4.1.1 An Infinite Line

As a simple model for electrodes with adsorbing sites, we consider now that a line of impurities is introduced into the plasma. We shall consider the case where the impurities are very

close to each other, so there is a continuous adsorbing line in the plasma, located at $x = x_0$. Since the system is translational invariant in the direction of the line (the y direction), it is better to work in Cartesian coordinates and confine the plasma in a rectangular box. The fugacity can be modeled by

$$m_\pm(\mathbf{r}) = m_\pm(x) = m\big[1 + \bar{\alpha}_\pm \delta(x - x_0)\big]. \tag{4.1}$$

The adhesivity $\bar{\alpha}_\pm$ has length dimensions, unlike the adhesivity $\alpha_\pm$ for point impurities which has area dimensions. Actually, $\bar{\alpha}_\pm$ is an effective adhesivity due to the distribution of the adsorbing particles on the line. We will assume that the line is composed of "positive" impurities, so $\bar{\alpha}_+ = \bar{\alpha}_1$ and $\bar{\alpha}_- = 0$.

Due to the translational invariance in the y direction, this problem is easier to solve via Fourier transform. For this purpose we define

$$G_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{2\pi} \int_{\mathbb{R}} \widetilde{G}_{s_1 s_2}(x_1, x_2, l) e^{il(y_1 - y_2)} dl. \tag{4.2}$$

By replacing (4.2) and the given fugacities into (2.2), we obtain the system

$$m\big[1 + \bar{\alpha}_1 \delta(x_1 - x_0)\big]\widetilde{G}_{++}(x_1, x_2, l)$$
$$+ \left(\frac{d}{dx_1} + l\right)\widetilde{G}_{-+}(x_1, x_2, l) = \delta(x_1 - x_2), \tag{4.3a}$$

$$\left(\frac{d}{dx_1} - l\right)\widetilde{G}_{++}(x_1, x_2, l) + m\widetilde{G}_{-+}(x_1, x_2, l) = 0, \tag{4.3b}$$

$$m\big[1 + \bar{\alpha}_1 \delta(x_1 - x_0)\big]\widetilde{G}_{+-}(x_1, x_2, l)$$
$$+ \left(\frac{d}{dx_1} + l\right)\widetilde{G}_{--}(x_1, x_2, l) = 0, \tag{4.3c}$$

$$\left(\frac{d}{dx_1} - l\right)\widetilde{G}_{+-}(x_1, x_2, l) + m\widetilde{G}_{--}(x_1, x_2, l) = \delta(x_1 - x_2). \tag{4.3d}$$

We assume solutions of the form $\widetilde{G}_{s_1 s_2} = \widetilde{G}^0_{s_1 s_2} + \widetilde{G}^1_{s_1 s_2}$ where $\widetilde{G}^0_{s_1 s_2}$ are the bulk solutions for $\bar{\alpha}_1 = 0$. It is easy to show that these solutions are given by [1]

$$\widetilde{G}^0_{\pm\pm}(x_1, x_2, l) = \frac{m}{2k} e^{-k|x_1 - x_2|}, \tag{4.4a}$$

$$\widetilde{G}^0_{-+}(x_1, x_2, l) = \frac{1}{m}\left(-\frac{d}{dx_1} + l\right)\widetilde{G}^0_{++}(x_1, x_2, l), \tag{4.4b}$$

$$\widetilde{G}^0_{+-}(x_1, x_2, l) = -\frac{1}{m}\left(\frac{d}{dx_1} + l\right)\widetilde{G}^0_{--}(x_1, x_2, l), \tag{4.4c}$$

with $k = \sqrt{m^2 + l^2}$.

The solutions to (4.3) are

$$\widetilde{G}_{++}(x_1, x_2, l) = \begin{cases} \frac{m}{2k} e^{-k|x_1 - x_2|} - \frac{m^3 \bar{\alpha}_1}{2k(2k + m^2 \bar{\alpha}_1)} e^{-k|2x_0 - x_1 - x_2|}, & x_0 < x_1, x_2, \\ & \text{or } x_0 > x_1, x_2, \\ \frac{m}{2k + m^2 \bar{\alpha}_1} e^{-k|x_1 - x_2|}, & x_0 \text{ between } x_1 \text{ and } x_2, \end{cases} \tag{4.5}$$

$$\widetilde{G}_{--}(x_1, x_2, l) = \begin{cases} \frac{m}{2k}e^{-k|x_1-x_2|} + \frac{m\bar{\alpha}_1(k+l)^2}{2k(2k+m^2\bar{\alpha}_1)}e^{k(2x_0-x_1-x_2)}, & x_0 < x_1, x_2, \\ \frac{m}{2k+m^2\bar{\alpha}_1}e^{-k|x_1-x_2|}, & x_0 \text{ between } x_1 \text{ and } x_2, \quad (4.6) \\ \frac{m}{2k}e^{-k|x_1-x_2|} + \frac{m\bar{\alpha}_1(k-l)^2}{2k(2k+m^2\bar{\alpha}_1)}e^{-k(2x_0-x_1-x_2)}, & x_0 > x_1, x_2. \end{cases}$$

The number density profiles n_+, n_-, and charge density profile ρ (in units of e) are obtained using (2.3). We find

$$n_+(x) = \left[1 + \bar{\alpha}_1\delta(x-x_0)\right]n_+^*(x), \tag{4.7a}$$

$$n_-(x) = n_-^*(x), \tag{4.7b}$$

$$\rho(x) = n_+(x) - n_-(x) = \bar{\alpha}_1\delta(x-x_0)n_+^*(x_0) + \rho^*(x), \tag{4.7c}$$

with

$$n_+^*(x) = n_+^0 - m^3\bar{\alpha}_1 \int_0^\infty \frac{dt}{4\pi} \frac{e^{-2\sqrt{t^2+1}m|x-x_0|}}{\sqrt{t^2+1}(\sqrt{t^2+1}+\tilde{\alpha})}, \tag{4.8a}$$

$$n_-^*(x) = n_-^0 + m^3\bar{\alpha}_1 \int_0^\infty \frac{dt}{4\pi} \frac{(1+2t^2)e^{-2\sqrt{t^2+1}m|x-x_0|}}{\sqrt{t^2+1}(\sqrt{t^2+1}+\tilde{\alpha})}, \tag{4.8b}$$

$$\rho^*(x) = -m^3\bar{\alpha}_1 \int_0^\infty \frac{dt}{2\pi} \frac{\sqrt{t^2+1}\,e^{-2\sqrt{t^2+1}m|x-x_0|}}{\sqrt{t^2+1}+\tilde{\alpha}}, \tag{4.8c}$$

with $\tilde{\alpha} = m\bar{\alpha}_1/2$.

Figure 8 show the charge density profile for this configuration. The curves obtained are similar to the ones obtained for point impurities. They differ in the magnitude as well as in the peaks near the adsorbing particles. The magnitude of the density at the adsorbing line is reduced as a consequence of the impurities distribution.
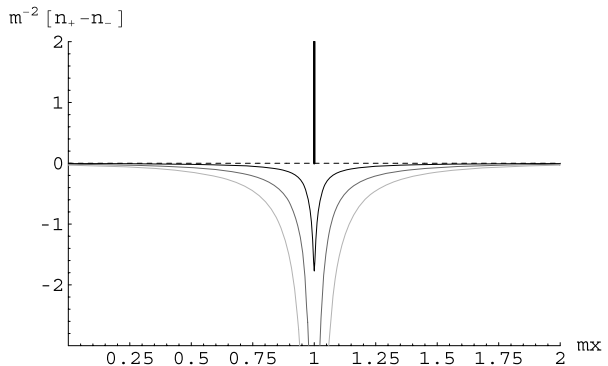
The line charge density adsorbed by the line, σ, can be computed by the relation

$$\sigma = \bar{\alpha}_1 n_+^*(x_0) \tag{4.9}$$

or by integrating the opposite of the nonadsorbed charge density

$$\sigma = -\int_{-\infty}^\infty \rho^*(x)\,dx. \tag{4.10}$$

Fig. 8 Charge density profile for a line of "positive" impurities at $mx = 1$. The adhesivities are $m\bar{\alpha}_1 = 0.2, 1$ and 3 from the *darkest* to the *lightest*. The cutoff is $ma = 0.01$

Either way, the result is the same, as expected,

$$\sigma = m^2 \bar{\alpha}_1 \int_0^{t_{max}} \frac{dt}{2\pi} \frac{1}{\sqrt{t^2 + 1} + \tilde{\alpha}} \tag{4.11a}$$

$$= \bar{\alpha}_1 n_+^0 - \frac{m\tilde{\alpha}^2}{\pi} \int_0^{\infty} \frac{dt}{t^2 + 1 + \tilde{\alpha}\sqrt{t^2 + 1}} \tag{4.11b}$$

$$= \bar{\alpha}_1 n_+^0 - \frac{m}{\pi} \frac{\tilde{\alpha}^2}{\sqrt{\tilde{\alpha}^2 - 1}} \cosh^{-1} \tilde{\alpha}, \tag{4.11c}$$

where we introduced an ultraviolet cutoff $t_{max} = e^{-C}/(ma)$ to obtain a finite result. This choice of the cutoff is done to ensure the correct result for bulk density [1]

$$n_+^0 = \frac{m^2}{2\pi} \int_0^{t_{max}} \frac{dt}{\sqrt{t^2 + 1}}. \tag{4.12}$$

It is worth mentioning that the integral in (4.11b) can be expressed in several different equivalent ways, by using known identities for the inverse trigonometric and hyperbolic functions [14],

$$\int_0^{\infty} \frac{dt}{t^2 + 1 + \tilde{\alpha}\sqrt{t^2 + 1}} = \int_0^{\pi/2} \frac{dx}{1 + \tilde{\alpha}\cos x} \tag{4.13a}$$

$$= \frac{2}{\sqrt{1 - \tilde{\alpha}^2}} \tan^{-1} \frac{\sqrt{1 - \tilde{\alpha}^2}}{1 + \tilde{\alpha}} \tag{4.13b}$$

$$= \frac{1}{\sqrt{1 - \tilde{\alpha}^2}} \tan^{-1} \frac{\sqrt{1 - \tilde{\alpha}^2}}{\tilde{\alpha}} \tag{4.13c}$$

$$= \frac{1}{\sqrt{\tilde{\alpha}^2 - 1}} \tanh^{-1} \frac{\sqrt{\tilde{\alpha}^2 - 1}}{\tilde{\alpha}} \tag{4.13d}$$

$$= \frac{\cosh^{-1} \tilde{\alpha}}{\sqrt{\tilde{\alpha}^2 - 1}}. \tag{4.13e}$$

In particular, use was made of the identity [14]

$$2\tan^{-1} z = \tan^{-1} \frac{2z}{1 - z^2}. \tag{4.14}$$

Also, it should be noticed that the result (4.11c) is valid for all values of $\tilde{\alpha} > 0$, even when $\tilde{\alpha} < 1$ there both the square root and the $\cosh^{-1}$ are imaginary but the total result is real.

Contrary to the case of a point adsorbing impurity, where the adsorbed charge reaches a maximum value of 1 when $\alpha_1 \to \infty$, the adsorbed line charge density is not bounded when $\bar{\alpha}_1 \to \infty$. Indeed, for large $\bar{\alpha}_1$,

$$\sigma = \bar{\alpha}_1 n_+^0 - \frac{m\tilde{\alpha}}{\pi} \left[\ln(2\tilde{\alpha}) + \frac{1}{2\tilde{\alpha}^2} \left[\ln(2\tilde{\alpha}) - \frac{1}{2} \right] + O(\tilde{\alpha}^{-4}) \right]. \tag{4.15}$$

It is instructive to compute the electric field $E(x)$ and electric potential $\phi(x)$ created by the adsorbed line charge density and its polarization cloud. Integrating Poisson equation we

obtain, for $x > 0$ and choosing $x_0 = 0$,

$$E(x) = em\tilde{\alpha} \int_0^\infty \frac{e^{-2\sqrt{t^2+1}mx} \, dt}{\sqrt{t^2+1}+\tilde{\alpha}} \tag{4.16}$$

and

$$\phi(x) = e\tilde{\alpha} \int_0^\infty \frac{e^{-2\sqrt{t^2+1}mx} \, dt}{t^2+1+\tilde{\alpha}\sqrt{t^2+1}}. \tag{4.17}$$

Notice that the electric field at the position of the impurity is related to the adsorbed line charge density by $E(0^+) = \pi\sigma$, and as usual we have the expected discontinuity $E(0^+) - E(0^-) = 2\pi\sigma$. The potential drop from the adsorbed line to infinity is

$$\phi(0) - \phi(+\infty) = e\tilde{\alpha} \int_0^\infty \frac{dt}{t^2+1+\tilde{\alpha}\sqrt{t^2+1}} = \frac{e\tilde{\alpha}}{\sqrt{\tilde{\alpha}^2-1}} \cosh^{-1}\tilde{\alpha}. \tag{4.18}$$

The correlation functions can be obtained from the Green functions (4.5) and (4.6), using the relation (2.3b). Let us simply note that the correlation functions have an exponential decay both in the x direction (transverse to the adsorbing line) and the y direction (along the adsorbing line), a behavior similar to the case without adsorbing line. This is different from the algebraic decay that the correlation functions show parallel along a hard wall [15–17].

4.1.2 Two Infinite Lines

For two parallel infinite lines it is also possible to find analytic expressions for the Green functions. For two lines located at $x = X_1$ and $x = X_2$, both attracting the positive particles, the fugacities are

$$m_+(x) = m\big[1 + \bar{\alpha}_1\delta(x - X_1) + \bar{\alpha}_2\delta(x - X_2)\big], \tag{4.19}$$

$$m_-(x) = m. \tag{4.20}$$

For one line, located at $x = X_1$, attracting the positive particles, and a second line, located at $x = X_2$, attracting the negative particles, the fugacities are

$$m_+(x) = m(1 + \bar{\alpha}_1\delta(x - X_1)), \tag{4.21}$$

$$m_-(x) = m(1 + \bar{\alpha}_2\delta(x - X_2)). \tag{4.22}$$

Without loss of generality, we suppose that $X_1 < X_2$. The resolution of the linear system (2.2) satisfied by the Green functions follows similar steps as for the previous case of one line. The explicit expressions for the Green functions depend on the relative positions of their arguments with respect to the lines. The complete expressions can be found in [13]. Let us focus here in the expressions obtained for the density profiles.

For two lines attracting positive particles, we find the positive, negative and charge densities

$$n_+(x) = \big[1 + \bar{\alpha}_1\delta(x - X_1) + \bar{\alpha}_2\delta(x - X_2)\big]n_+^*(x), \tag{4.23a}$$

$$n_-(x) = n_-^*(x), \tag{4.23b}$$

$$\rho(x) = \big[\bar{\alpha}_1\delta(x - X_1) + \bar{\alpha}_2\delta(x - X_2)\big]n_+^*(x) + \rho^*(x), \tag{4.23c}$$

with

$$n_+^*(x) = n_+^0 - \int_{-\infty}^{\infty} \frac{m^4 e^{-2kx}}{4\pi k} \frac{\bar{\alpha}_1(2k - m^2\bar{\alpha}_2)e^{2kX_1} + \bar{\alpha}_2(2k + m^2\bar{\alpha}_1)e^{2kX_2}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}} \, dl, \quad (4.24a)$$

$$n_-^*(x) = n_-^0 + \int_{-\infty}^{\infty} \frac{m^2(k+l)^2 e^{-2kx}}{4\pi k} \frac{\bar{\alpha}_1(2k - m^2\bar{\alpha}_2)e^{2kX_1} + \bar{\alpha}_2(2k + m^2\bar{\alpha}_1)e^{2kX_2}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}} \, dl,$$
(4.24b)

$$\rho^*(x) = -\int_{-\infty}^{+\infty} \frac{m^2 k e^{-2kx}}{2\pi} \frac{\bar{\alpha}_1(2k - m^2\bar{\alpha}_2)e^{2kX_1} + \bar{\alpha}_2(2k + m^2\bar{\alpha}_1)e^{2kX_2}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}} \, dl \quad (4.24c)$$

for $X_1 < X_2 < x$, with $k = \sqrt{m^2 + l^2}$. Between the lines, we find

$$n_+^*(x) = n_+^0 - \int_{-\infty}^{\infty} \frac{m^4}{4\pi k} \left[\frac{\bar{\alpha}_1(2k + m^2\bar{\alpha}_2)e^{2k(X_1 - x)}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}} \right.$$
$$\left. + \frac{\bar{\alpha}_2(2k + m^2\bar{\alpha}_1)e^{2k(x - X_2)} - 2m^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}} \right] dl \quad (4.25a)$$

$$n_-^*(x) = n_-^0 + \int_{-\infty}^{\infty} \frac{m^2}{4\pi k} \left[\frac{\bar{\alpha}_1(k+l)^2(2k + m^2\bar{\alpha}_2)e^{2k(X_1 - x)}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}} \right.$$
$$\left. + \frac{\bar{\alpha}_2(k-l)^2(2k + m^2\bar{\alpha}_1)e^{2k(x - X_2)} + 2m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}} \right] dl \quad (4.25b)$$

$$\rho^*(x) = -\int_{-\infty}^{+\infty} \frac{m^2 k}{2\pi} \frac{\bar{\alpha}_1(2k + m^2\bar{\alpha}_2)e^{2k(X_1 - x)} + \bar{\alpha}_2(2k + m^2\bar{\alpha}_1)e^{2k(x - X_2)}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1 - X_2)}} \, dl \quad (4.25c)$$

for $X_1 < x < X_2$. It is straightforward to verify that these expressions reduce to case of one adsorbing line when $X_1 \to -\infty$, as it should. Also, one can verify that for $X_1 = X_2$, we recover the density profiles for one line with adhesivity $\bar{\alpha}_1 + \bar{\alpha}_2$. Figure 9 shows the charge density profile in the presence of two positive lines.

The line density of adsorbed particles by each line, σ_1 and σ_2, are obtained from $\sigma_1 = \bar{\alpha}_1 n_+^*(X_1)$ and a similar expression for σ_2. Explicitly,

Fig. 9 Charge density profile as a function of the distance. The adhesivities are $m\bar{\alpha}_1 = 0.2$ and $m\bar{\alpha}_2 = 0.3$. The position of the lines are $mX_1 = 1$, $mX_1 = 1.3$ and $mX_2 = 2$, $mX_2 = 1.7$ from the *darkest* to the *lightest*. The cutoff is $ma = 0.01$
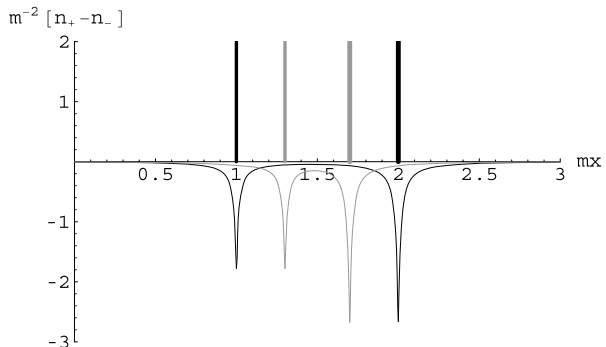
$$\sigma_1 = \bar{\alpha}_1 n_+^0 - \frac{m^4 \bar{\alpha}_1}{4\pi} \int_{-\infty}^{+\infty} \frac{[\bar{\alpha}_1(2k + m^2\bar{\alpha}_2) + \bar{\alpha}_2(2k - m^2\bar{\alpha}_1)e^{2k(X_1-X_2)}]\,dl}{k[(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}]}$$

$$= \frac{m^4}{2\pi} \int_{-\infty}^{+\infty} \frac{(2k + m^2\bar{\alpha}_2)\bar{\alpha}_1 - m^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}\,dl \tag{4.26}$$

and a similar expression for σ_2 interchanging $\bar{\alpha}_1$ and $\bar{\alpha}_2$. In the second line of (4.26) we used the formal expression

$$n_+^0 = \frac{m^2}{4\pi} \int_{-\infty}^{+\infty} \frac{dl}{k} \tag{4.27}$$

for the bulk density (actually, the integral should be cutoff to a $l_{max} = e^{-C}/a$ to obtain finite results). The total adsorbed charge is found to be

$$\sigma_1 + \sigma_2 = \frac{m^2}{\pi} \int_{-\infty}^{+\infty} \frac{(\bar{\alpha}_1 + \bar{\alpha}_2)k + m^2\bar{\alpha}_1\bar{\alpha}_2[1 - e^{2k(X_1-X_2)}]}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) - m^4\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}. \tag{4.28}$$

It is straightforward to verify that

$$\sigma_1 + \sigma_2 = -\int_{-\infty}^{+\infty} \rho^*(x)\,dx \tag{4.29}$$

as it should be, since the system is globally neutral.

In the case where line at X_2 adsorbs negative particles while the line at X_1 adsorbs the positive ones, we find the following density profiles

$$n_+(x) = \left[1 + \bar{\alpha}_1\delta(x - X_1)\right]n_+^*(x), \tag{4.30a}$$

$$n_-(x) = \left[1 + \bar{\alpha}_2\delta(x - X_2)\right]n_-^*(x), \tag{4.30b}$$

$$\rho(x) = \bar{\alpha}_1\delta(x - X_1)n_+^*(X_1) - \bar{\alpha}_2\delta(x - X_2)n_-^*(X_2) + \rho^*(x), \tag{4.30c}$$

with

$$n_+^*(x) = n_+^0 - \int_{-\infty}^{\infty} \frac{m^2 e^{-2kx}}{4\pi k} \frac{m^2\bar{\alpha}_1(2k - m^2\bar{\alpha}_2)e^{2kX_1} - \bar{\alpha}_2(k - l)^2(2k + m^2\bar{\alpha}_1)e^{2kX_2}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) + m^2\bar{\alpha}_1\bar{\alpha}_2(k + l)^2 e^{2k(X_1-X_2)}}\,dl, \tag{4.31a}$$

$$n_-^*(x) = n_-^0 - \int_{-\infty}^{\infty} \frac{m^2 e^{-2kx}}{4\pi k} \frac{\bar{\alpha}_1(k + l)^2(m^2\bar{\alpha}_2 - 2k)e^{2kX_1} + m^2\bar{\alpha}_2(2k + m^2\bar{\alpha}_1)e^{2kX_2}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) + m^2\bar{\alpha}_1\bar{\alpha}_2(k + l)^2 e^{2k(X_1-X_2)}}\,dl, \tag{4.31b}$$

$$\rho^*(x) = -\int_{-\infty}^{+\infty} \frac{m^2 e^{-2kx}}{2\pi} \frac{\bar{\alpha}_1(k + l)(2k - m^2\bar{\alpha}_2)e^{2kX_1} + \bar{\alpha}_2(l - k)(2k + m^2\bar{\alpha}_1)e^{2kX_2}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) + m^2(k + l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}\,dl \tag{4.31c}$$

for $X_1 < X_2 < x$. For x between the lines, $X_1 < x < X_2$, we find

$$n_+^*(x) = n_+^0 + \int_{-\infty}^{\infty} \frac{m^2}{4\pi k} \left[\frac{m^2\bar{\alpha}_1(-2k - m^2\bar{\alpha}_2)e^{2k(X_1-x)} + \bar{\alpha}_2(k + l)^2(2k + m^2\bar{\alpha}_1)e^{2k(x-X_2)}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) + m^2(k + l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}\right.$$

$$\left. - \frac{2m^2\bar{\alpha}_1\bar{\alpha}_2(k + l)^2 e^{2k(X_1-X_2)}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2) + m^2(k + l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}\right]dl, \tag{4.32a}$$

$$n_-^*(x) = n_-^0 + \int_{-\infty}^{\infty} \frac{m^2}{4\pi k} \left[\frac{\bar{\alpha}_1(k+l)^2(2k+m^2\bar{\alpha}_2)e^{2k(X_1-x)} - m^2\bar{\alpha}_2(2k+m^2\bar{\alpha}_1)e^{2k(x-X_2)}}{(2k+m^2\bar{\alpha}_1)(2k+m^2\bar{\alpha}_2) + m^2(k+l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}} \right.$$

$$\left. - \frac{2m^2\bar{\alpha}_1\bar{\alpha}_2(k+l)^2 e^{2k(X_1-X_2)}}{(2k+m^2\bar{\alpha}_1)(2k+m^2\bar{\alpha}_2) + m^2(k+l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}} \right] dl, \tag{4.32b}$$

$$\rho^*(x) = - \int_{-\infty}^{+\infty} \frac{m^2(k+l)}{2\pi} \frac{\bar{\alpha}_1(2k+m^2\bar{\alpha}_2)e^{2k(X_1-x)} - \bar{\alpha}_2(2k+m^2\bar{\alpha}_1)e^{2k(x-X_2)}}{(2k+m^2\bar{\alpha}_1)(2k+m^2\bar{\alpha}_2) + m^2(k+l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}} dl. \tag{4.32c}$$

The line charge density adsorbed by each line is now given by $\sigma_1 = \bar{\alpha}_1 n_+^*(X_1)$ and $\sigma_2 = -\bar{\alpha}_2 n_-^*(X_2)$,

$$\sigma_1 = \frac{\alpha_1 m^2}{2\pi} \int_{-\infty}^{+\infty} \frac{[2k+m^2\bar{\alpha}_2 + (k+l)^2\bar{\alpha}_2 e^{2k(X_1-X_2)}] dl}{(2k+m^2\bar{\alpha}_1)(2k+m^2\bar{\alpha}_2) + m^2(k+l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}, \tag{4.33a}$$

$$\sigma_2 = -\frac{\alpha_2 m^2}{2\pi} \int_{-\infty}^{+\infty} \frac{[2k+m^2\bar{\alpha}_1 + (k+l)^2\bar{\alpha}_1 e^{2k(X_1-X_2)}] dl}{(2k+m^2\bar{\alpha}_1)(2k+m^2\bar{\alpha}_2) + m^2(k+l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}. \tag{4.33b}$$

The total line charge density adsorbed by the lines is formally given by

$$\sigma_1 + \sigma_2 = (\bar{\alpha}_1 - \bar{\alpha}_2) \frac{m^2}{\pi} \int_{-\infty}^{+\infty} \frac{k\, dl}{(2k+m^2\bar{\alpha}_1)(2k+m^2\bar{\alpha}_2) + m^2(k+l)^2\bar{\alpha}_1\bar{\alpha}_2 e^{2k(X_1-X_2)}}. \tag{4.34}$$

Once again, it is straightforward to verify that

$$\sigma_1 + \sigma_2 = - \int_{-\infty}^{+\infty} \rho^*(x)\, dx \tag{4.35}$$

as imposed by the global neutrality of the system.

It is interesting to notice that in the two cases considered in this section, the density profile of particles that are adsorbed by one line is a continuous function as it crosses the line, while the density profile of the particles that are not adsorbed by this line is discontinuous when crossing such line. However, the discontinuity jump in this density is exponentially small for large separations between the two lines, it is of order $e^{-2m|X_1-X_2|}$.

4.2 Grand Potential

In the following subsections, we compute the grand potential of the plasma with one or two adsorbing lines at $\Gamma = 2$, by solving the eigenvalue problem (2.6).

4.2.1 An Infinite Line

For this case we confine the plasma into a rectangular box of area $2L_x \times L_y$ and we work in Cartesian coordinates. The boundaries and the adsorbing line, located at $x = 0$, divide the space in four regions $x < -L_x$, $-L_x < x < 0$, $0 < x < L_x$ and $x > L_x$ which will be labeled by (1), (2), (3) and (4).

The eigenvalue system (2.6) takes the form

$$m[1 + \bar{\alpha}_1\delta(x)]g(\mathbf{r}) = 2\lambda \partial_z f(\mathbf{r}), \tag{4.36a}$$

$$mf(\mathbf{r}) = 2\lambda \partial_{\bar{z}} g(\mathbf{r}) \tag{4.36b}$$

for $-L_x \leq x \leq L_x$. Outside that region, $m_+(x) = m_-(x) = 0$. Thus we conclude that for $x \notin [-L_x, L_x]$, $\partial_z f = 0$ and $\partial_{\bar{z}} g = 0$, that is f is an anti-analytic function of z and g is an analytic function of z.

The translation symmetry on the y axis allows us to assume $g(\mathbf{r})$ of the form $g(x, y) = \tilde{g}(x)e^{ily}$. The general solution for the four regions can be written as

$$g^{(1)}(x, y) = A^{(1)}e^{lx+ily}, \tag{4.37a}$$

$$g^{(2)}(x, y) = A^{(2)}e^{k_x x+ily} + B^{(2)}e^{-(k_x x-ily)}, \tag{4.37b}$$

$$g^{(3)}(x, y) = A^{(3)}e^{k_x x+ily} + B^{(3)}e^{-(k_x x-ly)}, \tag{4.37c}$$

$$g^{(4)}(x, y) = B^{(4)}e^{lx+ily}, \tag{4.37d}$$

and

$$f^{(1,4)}(x, y) = C^{(1,4)}e^{-lx+ily}, \tag{4.38a}$$

$$f^{(2,3)}(x, y) = \frac{\lambda}{m}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)g^{(2,3)}(x, y), \tag{4.38b}$$

with $k_x = \sqrt{l^2 + \frac{m^2}{\lambda^2}}$.

Since the eigenfunctions must vanish at $x = \pm\infty$, we conclude that $g^{(1)} = f^{(4)} = 0$ for $l < 0$ and $f^{(1)} = g^{(4)} = 0$ if $l > 0$. The boundary conditions demand that both $g(x, y)$ and $f(x, y)$ must be continuous at $x = \pm L_x$. At $x = 0$, $g(x, y)$ is continuous and $f(x, y)$ is discontinuous due to the Dirac distribution in (4.36a):

$$f(0^+, y) - f(0^-, y) = \frac{m\bar{\alpha}_1}{\lambda}g(0, y). \tag{4.39}$$

Assuming $l > 0$, these boundary conditions can be expressed by the linear system

$$\begin{pmatrix} (k_x - l)e^{-k_x L_x} & -(k_x + l)e^{k_x L_x} & 0 & 0 \\ 0 & 0 & e^{k_x L_x} & e^{-k_x L_x} \\ 1 & 1 & -1 & -1 \\ m^2\bar{\alpha}_1 - \lambda^2(k_x - l) & m^2\bar{\alpha}_1 + \lambda^2(k_x + l) & \lambda^2(k_x - l) & -\lambda^2(k_x + l) \end{pmatrix}\begin{pmatrix} A_2 \\ B_2 \\ A_3 \\ B_3 \end{pmatrix} = 0. \tag{4.40}$$

The functions $g(x, y)$ and $f(x, y)$ must not be zero, hence we demand that the determinant of the latter matrix must vanish. This condition leads us to the relation

$$\cosh(2k_x L_x)\left[1 + \frac{m^2\bar{\alpha}_1 l\lambda^{-2}}{2k_x^2}\right] + \sinh(2k_x L_x)\left[\frac{l}{k_x} + \frac{m^2\bar{\alpha}_1\lambda^{-2}}{2k_x}\right] + \frac{m^2\bar{\alpha}_1 l\lambda^{-2}}{2k_x^2} = 0. \tag{4.41}$$

For each value of l, there are several possible solutions to (4.41) for the eigenvalue λ, which we will denote as $\{\lambda_{l,n}\}_n$. For $l < 0$, we obtain (4.41) changing l by $-l$. To obtain the grand potential, we can recognize its relationship with a Weierstrass product [19] as follows. Let us define the analytic function, for $l \geq 0$,

$$h_l(z) = \left(1 + \frac{m^2\bar{\alpha}_1 l z^2}{2(m^2 z^2 + l^2)}\right)\cosh\left(2\sqrt{m^2 z^2 + l^2}L_x\right) + \frac{m^2\bar{\alpha}_1 l z^2}{2(m^2 z^2 + l^2)}$$

$$+ \left(\frac{l}{\sqrt{l^2 + m^2 z^2}}\right)\left(1 + \frac{m^2\bar{\alpha}_1 z^2}{2l}\right)\sinh\left(2\sqrt{m^2 z^2 + l^2}L_x\right). \tag{4.42}$$

We have $h_l(0) = e^{2lL_x}$, $h_l'(0) = 0$ and $h_l(z) = h_l(-z)$. By construction, the zeros of h_l are precisely the inverse of the eigenvalues $1/\lambda_{l,n}$. The representation of $h_l(z)$ as a Weierstrass infinite product is therefore

$$h_l(z) = h_l(0) \prod_n (1 - z\lambda_{l,n}). \tag{4.43}$$

Then, we notice that the grand potential is simply

$$\beta\Omega = -\sum_l \ln \prod_n (1 + \lambda_{l,n}) = -\frac{L_y}{\pi} \int_0^\infty dl \, \ln[e^{-2lL_x} h_l(-1)], \tag{4.44}$$

where we replaced the sum over l by an integral in the thermodynamic limit $L_y \to \infty$. In the thermodynamic limit, we also consider that $L_x \to \infty$, and we approximate $\cosh(2kL_x) \sim \sinh(2kL_x) \sim e^{2kL_x}/2$, with $k = \sqrt{l^2 + m^2}$. The final result is

$$\beta\Omega = \beta\Omega_0 + \beta\Omega_{\text{exc}}(\bar{\alpha}_1), \tag{4.45}$$

with

$$\beta\Omega_{\text{exc}}(\bar{\alpha}_1) = -\frac{L_y}{\pi} \int_0^{+\infty} \ln\left[1 + \frac{m^2\bar{\alpha}_1}{2\sqrt{m^2 + l^2}}\right] dl \tag{4.46}$$

and Ω_0 the grand potential of the unperturbed system for this geometry,

$$\Omega_0 = -2L_x L_y p_b + 2L_y \gamma + \mathcal{O}(e^{-2mL_x}) \tag{4.47}$$

with the bulk pressure p_b and the surface tension near an impenetrable wall γ,

$$\beta p_b = \frac{m^2}{2\pi}\left[\ln\frac{2}{ma} - C + \frac{1}{2}\right] = n_0 + \frac{m^2}{4\pi}, \tag{4.48}$$

$$\beta\gamma = m\left[\frac{1}{4} - \frac{1}{2\pi}\right]. \tag{4.49}$$

Hence we can conclude that the effective chemical potential of a line of length L_y and adhesivity $\bar{\alpha}$ is $\mu(\bar{\alpha}) = \Omega_{\text{exc}}^\pm(\bar{\alpha})$. The integral (4.46) can be computed explicitly. Actually, one must introduce an ultraviolet cutoff $l_{\max} = e^{-C}/a$ to obtain a finite result. This choice of the cutoff is done to ensure that the result for bulk pressure (4.48) is the same as the one obtained by integrating the bulk densities (3.7) [1]. Neglecting terms that vanish when $a \to 0$, we have

$$\beta\Omega_{\text{exc}}(\bar{\alpha}) = -\frac{mL_y}{\pi}\left[\tilde{\alpha}\ln\frac{2e^{-C}}{ma} - \sqrt{\tilde{\alpha}^2 - 1}\cosh^{-1}\tilde{\alpha} - \frac{\pi}{2} + \tilde{\alpha}\right], \tag{4.50}$$

with $\tilde{\alpha} = m\bar{\alpha}/2$. It is interesting to compare this result with the surface tension obtained in a similar problem where the plasma is confined in a strip with adsorbing boundaries [10]. In that problem the adsorbing line was located at a hard wall boundary. In our notations, the contribution of both the hard wall and the adsorbing line was [10]

$$\beta\Omega_{\text{exc}}^{\text{adsorbing wall}} = -\frac{mL_y}{4\pi}\left[m\bar{\alpha}\ln\frac{2e^{-C}}{ma} + 1 - \pi + m\bar{\alpha} + \frac{1 - (m\bar{\alpha})^2}{\bar{\alpha}}\ln(1 + m\bar{\alpha})\right]. \tag{4.51}$$

We notice that $\Omega_{\mathrm{exc}}(\bar{\alpha}) + L_y\gamma$ of our present problem is different from $\Omega_{\mathrm{exc}}^{\mathrm{adsorbing\,wall}}$. This is to be expected since each situation is different. In our present problem the adsorbing line is very far from the hard wall boundaries, while in [10], the adsorbing line is on the boundary itself. We notice however that the (divergent) dominant term, when the cutoff a vanish, for our problem is twice the one for the adsorbing boundary

$$\Omega_{\mathrm{exc}}(\bar{\alpha}) \sim 2\Omega_{\mathrm{exc}}^{\mathrm{adsorbing\,wall}} \sim \frac{mL_y}{2\pi}\ln(ma). \tag{4.52}$$

The factor 2 between both expressions can be understood if we realize that in our present problem the plasma is on both sides of the adsorbing line, while in [10] it is only on one side, the other side is empty.

From (4.46) and (4.11a) or from (4.50) and (4.11c), one can check that the relation between the adsorbed charge density σ and the excess grand potential

$$\sigma = -\bar{\alpha}_1 \frac{\partial(\beta\Omega/L_y)}{\partial\bar{\alpha}_1} \tag{4.53}$$

is satisfied.

4.2.2 Two Infinite Lines

Let us now consider the case where there are two infinite parallel adsorbing lines which attract the same kind of particles, say positive. The eigenvalue problem which must be solved is

$$mf(\mathbf{r}) = 2\lambda\partial_{\bar{z}}g(\mathbf{r}), \tag{4.54a}$$

$$m\big[1 + \bar{\alpha}_1\delta(x - X_1) + \bar{\alpha}_2(x - X_2)\big]g(\mathbf{r}) = 2\lambda\partial_z f(\mathbf{r}). \tag{4.54b}$$

We have placed the first line at $x = X_1$ and the second one at $x = X_2$. The method of solution is similar to the one for one line. Following similar steps as in the previous section, we find after some algebra [13]

$$\beta\Omega = \beta\Omega_0 + \beta\Omega_{\mathrm{exc}}(\bar{\alpha}_1) + \beta\Omega_{\mathrm{exc}}(\bar{\alpha}_2) + \beta\Omega^{\pm\pm}, \tag{4.55}$$

with

$$\beta\Omega^{\pm\pm} = -\frac{L_y}{\pi}\int_0^{+\infty}\ln\left[1 - \frac{m^4\bar{\alpha}_1\bar{\alpha}_2 e^{-2k|X_1-X_2|}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2)}\right]dl \tag{4.56}$$

and $\beta\Omega_{\mathrm{exc}}(\bar{\alpha})$ given in (4.46). We recall that $k = \sqrt{m^2 + l^2}$.

When the lines attract different types of particles the grand potential is

$$\beta\Omega = \beta\Omega_0 + \beta\Omega_{\mathrm{exc}}(\bar{\alpha}_1) + \beta\Omega_{\mathrm{exc}}(\bar{\alpha}_2) + \beta\Omega^{\pm\mp}, \tag{4.57}$$

with

$$\beta\Omega^{\pm\mp} = -\frac{L_y}{2\pi}\int_{-\infty}^{+\infty}\ln\left[1 + \frac{m^2\bar{\alpha}_1\bar{\alpha}_2(k+l)^2 e^{-2k|X_1-X_2|}}{(2k + m^2\bar{\alpha}_1)(2k + m^2\bar{\alpha}_2)}\right]dl. \tag{4.58}$$

The effective interaction between two lines (depending on their sign) is given by the expressions (4.56) and (4.58).

Fig. 10 Effective repulsion between two "positive" lines as a function of their distance $2x_0 = |X_1 - X_2|$. The adhesivities are $m\bar{\alpha}_1 = m\bar{\alpha}_2 = 0.2, 0.5, 1$ from the *darkest* to the *lightest*. The cutoff is $ma = 0.01$
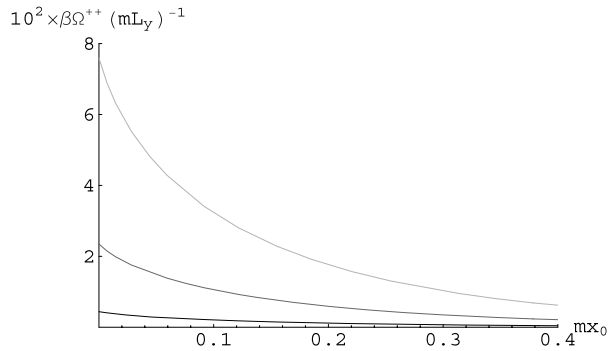
Fig. 11 Effective attraction between a positive and a negative line as a function of their distance $2x_0 = |X_1 - X_2|$. The values for the different parameters are the same as in Fig. 10
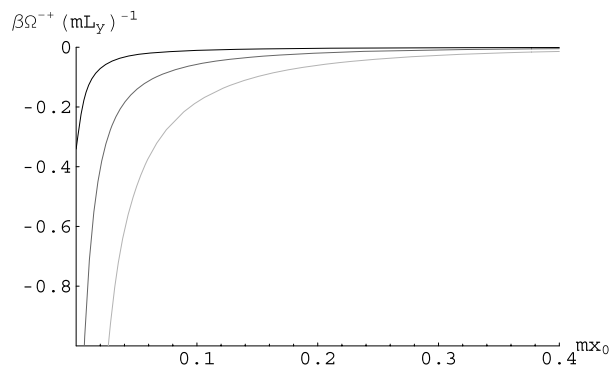
Figures 10 and 11 show the effective interaction between two infinite lines. As expected, like in the problem for point impurities, there is a repulsion if the sign of the lines is equal and a repulsion if it is different.

Notice that in both cases, $s = \pm$, $s' = \pm$, in the expression of the full excess grand potential $\Omega_{\text{exc}}(\bar{\alpha}_1) + \Omega_{\text{exc}}(\bar{\alpha}_2) + \Omega^{ss'}$ appears the logarithm of a term which is precisely the denominator in the Green functions and the density profiles (4.24), (4.25), (4.31) and (4.32). Also, one can easily check that the adsorbed charge on each line, computed in (4.26) and (4.33), can also be obtained from

$$\sigma_i = -\frac{\beta}{L_y}\alpha_i\frac{\partial}{\partial\alpha_i}\left(\Omega_{\text{exc}}(\bar{\alpha}_1) + \Omega_{\text{exc}}(\bar{\alpha}_2) + \Omega^{ss'}\right) \qquad (4.59)$$

for $i = 1, 2$.

5 Summary

In the present document we analyzed the behavior of the two-component plasma at $\Gamma = 2$ in the presence of one and two point adsorbing impurities, or in the presence of one and two adsorbing lines.

For point impurities, as shown in [8], the partition function and correlations of the system can be expressed in terms of the same quantities for an unperturbed system, without impurities. Since at $\Gamma = 2$, exact results are available for the partition function and correlations

of the unperturbed system, we were able to obtain exact results for the partition function, the density profiles and correlation functions of the plasma with one or two point impurities. We also computed the electric potential created by one impurity, due to the charge it adsorbs and the polarization cloud that is formed around it.

As a simple model for electrodes with adsorbing sites, we studied the properties of the plasma with one or two parallel lines of absorbing impurities. The general formalism developed in [1] for the two-component plasma with an external potential is applied to this case. We obtained exact results for the partition function and density profiles of the plasma. In both cases we checked that various relations between the adsorbed charge and the excess grand potential are satisfied.

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